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Universality and asymptotics in the Asymmetric Simple Exclusion Process

Marko Korhonen
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Ohjaaja: Antti Kupiainen

Tarkastajat: Antti Kupiainen
Kari Rummukainen

HELSINGIN YLIOPISTO
FYSIKAN LAITOS

PL 64 (Gustaf Hällströmin katu 2)
00014 Helsingin yliopisto

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<p>The purpose of this thesis is to study the asymmetric simple exclusion process, its asymptotics and some connections to other stochastic models. The text begins by giving some results on random matrix theory, such as the distribution function of the largest eigenvalue of a given random matrix.</p> <p>This is followed by a short section on the totally asymmetric simple exclusion process, which is a stochastic model of fermionic particles jumping only in one direction on a one-dimensional lattice. The probability that a given particle has jumped m times is then shown to be equal to the distribution of the largest eigenvalue of a specific type of a random matrix analyzed earlier.</p> <p>As this hints at some kind of universality, the particle model is then generalized to the asymmetric simple exclusion process, in which the particles can jump left or right. It turns out this model does not have the simple determinantal structure the earlier models had. The asymptotics of the model will then be analyzed, and it turns out there is a large universality class that encompasses all the models analyzed in the text.</p> <p>The reader is expected to be familiar with basic measure theory and complex analysis.</p>			
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Notation

The following listing is intended to clear any confusion arising from the notation in the text.

- \mathcal{H} – A separable Hilbert Space
- $\mathcal{L}(\mathcal{H})$ – The space of linear operators over \mathcal{H} .
- \mathcal{J}_1 – The space of trace-class operators.
- \mathcal{J}_2 – The space of Hilbert-Schmidt operators.
- $T^\otimes(V)$ – The tensor algebra of a vector space V .
- $\Lambda^\wedge(V)$ – The exterior algebra of a vector space V .
- $V^{\otimes k}$ – The tensor product of V with itself k times.
- $V^{\wedge k}$ – The wedge product of V with itself k times.
- $[N]$ – For parameters p and q , $[N] = \frac{p^N - q^N}{p - q}$.
- $[N]!$ – The q -factorial $[N]! = [N][N - 1] \cdots [1]$.
- $\begin{bmatrix} N \\ m \end{bmatrix}$ – The q -binomial defined by $\begin{bmatrix} N \\ m \end{bmatrix} = \frac{[N]!}{[N - m]![m]}$.

1 Introduction

The study of systems with a macroscopic amount of degrees of freedom has progressed surprisingly little in the past century. Despite several hundred years of development, modern mathematical and physical tools are not nearly powerful enough to exactly and efficiently solve a general enough system of more than three particles. The scientific theory of macroscopic systems consisting of a large number of microscopic constituents can be said to have been born in the 18th century with the advent of thermodynamics, often associated with names such as Daniel Bernoulli and Benjamin Thompson. Any serious treatment of the theory, however, was not done until the 19th century, after and during which the study of macroscopic equilibrium systems grew into a huge field with sciences such as chemistry branching off of it.

Branching out of classical thermodynamics grew statistical thermodynamics, or statistical mechanics as is the modern name. Statistical mechanics is the science of thermodynamic properties of systems through the study of their microscopic constituents. The statistical mechanics of systems in equilibrium is well established, culminating in the *fundamental postulate of equilibrium statistical mechanics*:

For an isolated system in equilibrium, the probability of finding it in some microstate is the same for all accessible microstates.

This result is realized through the microcanonical ensemble, which assigns an equal probability $\frac{1}{|\Omega|}$ to each microstate, where Ω is the set of all accessible microstates. Other such ensembles exist to characterize systems in equilibrium, such as the canonical ensemble which is used for systems which can exchange energy with the surroundings. A good introductory text on equilibrium statistical mechanics is [1].

Rigorously these ensembles are defined through the probability measures on the phase space of the system. From a physical point of view, the thermodynamic observables are time averages of the rapidly changing values of quantities in question, such as pressure and temperature. That these time averages equal the ensemble averages is called the *ergodicity hypothesis*. In other words, an ergodic system explores all points of its space over a long period of time.

The above discussion only holds for equilibrium systems. The field of non-equilibrium statistical mechanics – the study of systems potentially far from equilibrium – can be said to be still in its infancy, for there exist no equivalent results as there exist for equilibrium systems. Ergodicity of the systems is what made the equilibrium case so easy, for through that the ensemble averages could be used for calculation of the macroscopic observables. When out of equilibrium, time and space averages are not in general equivalent.

As such, non-equilibrium statistical mechanics today is considered one of the largest open fields in physics, and has experienced huge growth in the past few decades. As always, surprising amounts of understanding can be derived from the simplest of models. In some cases, they can even be used to model real systems fairly well. Since anything is rarely in any real equilibrium, the study of non-equilibrium is of interest in many other sciences as well. This includes, for example, biology, medicine and sociology.

Ideally, in the study non-equilibrium systems, a sample model has to be simple enough to allow exact solutions yet still yield non-trivial results. One such model was introduced in 1970 by F. Spitzer [2] in a paper studying interaction of Markov processes. This model is known as the Simple Exclusion Process (SEP) and is today considered a paradigmatic model in non-equilibrium statistical mechanics [3]. SEP is a stochastic process of interacting particles moving on a lattice, with interaction described by the exclusion property, meaning two particles cannot occupy the same site. It is very similar to a Heisenberg XXZ spin- $\frac{1}{2}$ chain [4], and it turns out both models are solvable by Bethe Ansatz [5], as

will be done for a special case of SEP in section 4. SEP is a Markov process, which makes the analysis much easier with the well established framework for Markov processes already existing.

There are several variants of the model. Asymmetric simple exclusion process (ASEP) is a lattice gas driven by an uniform external field, causing a bias of the particle current in either direction. The non-driven case is called the symmetric simple exclusion process (SSEP) and will not be explicitly studied here. ASEP will be defined and the dynamics analyzed in section 4.

It has been proved that ASEP, at least for some initial conditions, belongs to a class of models called the KPZ (Kardar-Parisi-Zhang) universality class. Universality is the independence of the properties of the system on the microscopic dynamics. The models need not be at all similar; In sections 3 and 4 it will be shown that the largest eigenvalue statistics for a type of random matrix are governed by the same equations as the transition probability of a special case of ASEP, the Totally Asymmetric Simple Exclusion Process (TASEP). A large class of more complicated models exhibiting the same kind of universality can be studied through ASEP owing to the equivalence of some macroscopic properties. In the case of ASEP it is the height function fluctuations of order $t^{\frac{1}{3}}$ and spatial correlations on the scale $t^{\frac{2}{3}}$ that are characteristic to its universality class. The models also usually have the same kind of limiting processes.

It is not the intention to present new and exciting proofs for the theorems contained in the text. The theorems will be stated as in the original papers, will be cited, and the proofs will be presented possibly modified by the author in an attempt to correct possible errors and make the presentation clearer, if needed, hopefully making the subject more accessible to a larger audience.

2 Preliminaries

In the text, a certain infinite determinant will often be used. The usual goal is to turn a series of multiple integrals into a Fredholm determinant of a trace-class operator. Most of the properties of finite-dimensional determinants hold, making further analysis easier.

2.1 Fredholm Determinants

As in the finite case, the trace of an operator is closely related to the determinant. Only trace-class operators (defined later) will be considered here. For a reader with a desire for a more detailed section on functional analysis, see [6] and [7]. Let $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \cdot \rangle$ be the standard inner product in \mathcal{H} . Define an inner product in $\mathcal{H}^{\otimes n}$ by

$$\langle \phi_1 \otimes \dots \otimes \phi_n, \psi_1 \otimes \dots \otimes \psi_n \rangle = \prod_{j=1}^n \langle \phi_j, \psi_j \rangle.$$

Assume $\{\phi_j\}_{j=1}^{\infty}$ is any orthonormal basis of \mathcal{H}

Definition (Trace class)

For a separable Hilbert space \mathcal{H} and a bounded linear operator A over \mathcal{H} such that $\langle \phi_j, A\phi_j \rangle \geq 0$ for all j (i.e. A is a positive operator), A is trace-class if and only if

$$\sum_{j=1}^{\infty} \langle \phi_j, A\phi_j \rangle < \infty,$$

the series converges absolutely and is independent of the choice of ϕ_j . For such operators, write $A \in \mathcal{J}_1(\mathcal{H})$.

Definition (Trace of a bounded linear operator)

Let \mathcal{H} be a separable Hilbert space. Take A to be a bounded linear operator over \mathcal{H} in the trace class. The trace of A is defined by

$$Tr(A) = \sum_{j=1}^{\infty} \langle \phi_j, A\phi_j \rangle.$$

This obviously converges by definition since A is in the trace class.

Define a norm in \mathcal{J}_1 by $\|A\|_1 = Tr(|A|)$ where $|A| = (A^*A)^{\frac{1}{2}}$.

Definition (Trace of a bounded multilinear operator)

Assume A and \mathcal{H} are as above. As shown in [6], $\|A^{\wedge k}\|_1 \leq \frac{\|A\|_1^k}{k!}$ and $A^{\wedge k} \in \mathcal{J}_1(\mathcal{H}^{\wedge k})$. The trace of $A^{\wedge n}$ then makes sense when defined by

$$Tr(A^{\wedge n}) = \sum_{1 \leq j_1 < \dots < j_n \leq \infty} \langle \phi_{j_1} \wedge \dots \wedge \phi_{j_n}, (A^{\wedge n})(\phi_{j_1} \wedge \dots \wedge \phi_{j_n}) \rangle.$$

Definition (Fredholm Determinant)

Let A be trace class. Then the Fredholm determinant of A is defined by

$$\det(id + A) = \sum_{k=0}^{\infty} Tr(A^{\wedge k}).$$

This converges since $|\det(I + A)| \leq \sum_{k=0}^{\infty} \|A^{\wedge k}\|_1 \leq e^{\|A\|_1}$

In addition to the basic definitions, one should be familiar with the Fredholm series expansion for the determinant. All the important properties and convergence arguments of the determinant, including the proposition below, can be found with proofs in [6].

Proposition 2.1 (*Fredholm Series Expansion*)

Let $\mathcal{H} = L^2$. Let A be trace class and $A(z_i, z_j)$ the integral kernel of A . Then, for any $\lambda \in \mathbb{C}$

$$\det(id + \lambda A) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_{\mathbb{R}^k} \det(A(z_i, z_j))_{i,j=1,\dots,k} dz_1 \cdots dz_k.$$

3 Random matrix theory

Random matrix theory was introduced in 1950s by Wigner and Dyson to model heavy atomic nuclei. It can be expected that given an atomic nucleus, it is in a state that shares properties with other similar systems. This can be generalized to any kind of system having a particular symmetry, and all these systems then combined in a statistical ensemble. This approach does not take into account microscopic properties characteristic to a single system, but it is assumed these properties cannot be observed in a large enough system, such as the heavier atomic nuclei.

The idea is to represent the Hamiltonian of the system by a random matrix with some specific symmetries depending on the system. By studying the spectral properties of ensembles of random matrices, one can obtain information about the energy spectrum of the physical system. The purpose of this analysis is not to obtain detailed information about the system, but describe the statistical behaviour in a

more general manner, such as finding the distribution of eigenvalue spacing, corresponding to energy level spacing distribution.

Heavy nuclei are prime examples of quantum mechanical systems exhibiting chaos, and thus relevant observables include the level spacing distribution [8]. Since eigenvalues of the Hamiltonian are the energy levels, the equivalent quantity here is the spacing of eigenvalues. In chaotic systems the eigenvalues exhibit mutual repulsion, so the usual Poissonian level spacing distribution of integrable systems cannot exist. Instead, e.g. in Gaussian Unitary Ensemble, the level spacing distribution approaches the Tracy-Widom distribution as the matrix size increases [8], as can be seen by analysing the relevant random matrix ensemble.

Obviously arbitrary random numbers cannot be used to describe an interesting physical system; In addition to the obvious independence and normalization arguments some additional assumptions on the matrices are required. The most common and well-known are the three Gaussian ensembles of matrices, which are as follows [9]:

1. Gaussian Orthogonal Ensemble (GOE)
Real symmetric matrices used to describe time reversal symmetric systems with even spin.
2. Gaussian Symplectic Ensemble (GSE)
Real quaternionic matrices used to describe time reversal symmetric systems with odd spin and no rotational invariance.
3. Gaussian Unitary Ensemble (GUE)
Complex Hermitian matrices used to describe systems without time reversal symmetry.

A classical source on random matrix theory is [9], so it is not necessary to go through it all in this paper. Instead, only the Gaussian Unitary and the Wishart Complex Ensembles will be introduced to show a remarkable connection of random matrices to other statistical models.

3.1 Beta Ensembles

There is a large class of random matrix ensembles called Beta Ensembles which encompass the Gaussian ensembles as well. The two investigated here are the Gaussian and Wishart ensembles for $\beta = 2$.

3.1.1 Density of Eigenvalues

Definition (Gaussian Unitary Ensemble)

The Gaussian Unitary Ensemble is defined on a space of $N \times N$ complex random hermitian matrices H as the probability measure

$$d\mathbb{P}_G^{(N)}(H) = \frac{1}{Z_{N,G}} e^{-\frac{1}{2N} \text{Tr}(H^2)} \prod_{i=1}^N dH_{i,i} \prod_{1 \leq i < j \leq N} d\text{Re}(H_{i,j}) d\text{Im}(H_{i,j}) \quad (3.1)$$

where Z'_N is a normalization constant. This coincides with the third entry in the list above. To be more specific, this form is actually called the Wigner Ensemble and one obtains GUE by $\frac{1}{2N} \rightarrow \frac{1}{2}$. [9]

Definition (Wishart Complex Ensemble)

The Wishart Complex Ensemble (WCE) is defined on the space of matrices $H = AA^\dagger$, where A is a complex $N \times M$ Gaussian random matrix, as the probability measure

$$d\mathbb{P}_W^{(N,M)}(H) = \frac{1}{Z_{N,M,W}} (\det(H))^{N-M} e^{-\frac{\text{Tr}(H)}{2}} \prod_{i=1}^N dA_{i,i} \prod_{1 \leq i < j \leq N} d\text{Re}(A_{i,j}) d\text{Im}(A_{i,j}) \quad (3.2)$$

where $Z'_{N,M,W}$ is a normalization constant.

Define $\omega(H)_X^{(N,M)}$ by

$$\omega_X^{(N,M)}(H) = \begin{cases} \frac{1}{Z_{N,G}} e^{-\frac{1}{2N} \text{Tr}(H^2)} & \text{if } X = G \\ \frac{1}{Z_{N,M,W}} (\det(H))^{N-M} e^{-\frac{\text{Tr}(H)}{2}} & \text{if } X = W. \end{cases} \quad (3.3)$$

The goal now is to find the eigenvalue densities $dP_X^{(N,M)}(\lambda_1, \dots, \lambda_N)$ from equations (3.1) and (3.2). This enables easier analysis of some interesting properties of the eigenvalues, e.g. the distribution of the largest eigenvalue or the level spacing distribution. A complex hermitian $N \times N$ matrix has at most N^2 independent entries and N simple (unique) eigenvalues, meaning the uninteresting $N(N-1)$ variables will need to be integrated out. The content of theorem 3.1 below is essentially the separation of these variables from the eigenvalues.

Let $\mathbb{H} \subset GL(N, \mathbb{C})$ be the space of all complex $N \times N$ hermitian matrices. Then, for $H \in \mathbb{H}$,

$$H = \Gamma \Lambda \Gamma^\dagger \quad (3.4)$$

for some unitary Γ and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$, where $\{\lambda_i\}$ is the set of eigenvalues of H . Obviously $\Lambda = \Lambda(\lambda)$ depends only on the eigenvalues, while $\Gamma = \Gamma(p)$ is some function of $N(N-1)$ variables p_i , which are all the independent variables left after the eigenvalues.

Define $\phi : \mathbb{H} \rightarrow \mathbb{H}$ as the coordinate transform $\phi(H) = (\Gamma^\dagger H \Gamma, \Gamma) = (\Lambda, \Gamma)$ for Γ and Λ as in (3.4).

Theorem 3.1 (*Separation of eigenvalues*)

Let $H \in \mathbb{H}$ and ϕ , Λ and Γ defined as above. Then, for $dH = \prod_{i=1}^N dH_{i,i} \prod_{1 \leq i < j \leq N} d\text{Re}(H_{i,j}) d\text{Im}(H_{i,j})$, the Jacobian of the transform $H \mapsto \phi(H)$ is

$$J(\lambda_1, \dots, \lambda_N, p_1, \dots, p_{N(N-1)}) = g(p_1, \dots, p_{N(N-1)}) \prod_{i < j} |\lambda_j - \lambda_i|^2 \quad (3.5)$$

where g is some function of the variables p_i .

Proof As $J(\phi) = \left| \det \left(\frac{\partial H}{\partial \lambda_1}, \dots, \frac{\partial H}{\partial \lambda_N}, \frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_{N(N-1)}} \right) \right|$, one needs to calculate $\frac{\partial H}{\partial \lambda_i}$ and $\frac{\partial H}{\partial p_i}$. It can be seen that

$$\begin{aligned} \frac{\partial H(\lambda, p)}{\partial \lambda_i} &= \Gamma(p) \frac{\partial \Lambda(\lambda)}{\partial \lambda_i} \Gamma^\dagger(p) \\ \text{where } \left(\frac{\partial \Lambda(\lambda)}{\partial \lambda_i} \right)_{jk} &= \frac{\partial \lambda_j}{\partial \lambda_i} \delta_{jk} = \delta_{ij} \delta_{jk} \end{aligned}$$

Then, to calculate $\frac{\partial H(\lambda, p)}{\partial p_i}$, define $S_i = \Gamma^\dagger \frac{\partial \Gamma}{\partial p_i}$, so that $S_i^\dagger = \frac{\partial \Gamma^\dagger}{\partial p_i} \Gamma$. Use the fact that $\Gamma \Gamma^\dagger = \mathbb{I}$ and

$$\frac{\partial \mathbb{I}}{\partial p_i} = \frac{\partial \Gamma}{\partial p_i} \Gamma^\dagger + \Gamma \frac{\partial \Gamma^\dagger}{\partial p_i} = 0 = S_i + S_i^\dagger$$

so $S_i = -S_i^\dagger$. Using this one finally gets

$$\frac{\partial H}{\partial p_i} = \frac{\partial \Gamma}{\partial p_i} \Lambda \Gamma^\dagger + \Gamma \Lambda \frac{\partial \Gamma^\dagger}{\partial p_i} = \Gamma S_i \Lambda \Gamma^\dagger + \Gamma \Lambda S_i^\dagger \Gamma^\dagger = \Gamma S_i \Lambda \Gamma^\dagger - \Gamma \Lambda S_i \Gamma^\dagger = \Gamma [S_i, \Lambda] \Gamma^\dagger$$

where $[S_k, \Lambda] = S_k \Lambda - \Lambda S_k = (S_k)_{ij}(\lambda_j - \lambda_i)$ is the commutator. The map $H \xrightarrow{A} \Gamma^\dagger H \Gamma$ is obviously an unitary map for $A : \mathbb{H} \rightarrow \mathbb{H}$ and thus $\det(A) = \pm 1$. It then follows from the above calculations that

$$\left| \det(A(\frac{\partial H}{\partial \lambda_1}, \dots, \frac{\partial H}{\partial \lambda_N}, \frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_{N(N-1)}})) \right| = \left| \det(\frac{\partial \Lambda}{\partial \lambda_1}, \dots, \frac{\partial \Lambda}{\partial \lambda_N}, [S_1, \Lambda], \dots, [S_{N(N-1)}, \Lambda]) \right|$$

From the properties of the determinant, it follows that $\det(AH) = \det(\Gamma^\dagger H \Gamma) = \det(H)$. Thus the Jacobian of the transform ϕ is

$$\begin{aligned} J(\phi) &= \left| \det(\frac{\partial \Lambda}{\partial \lambda_1}, \dots, \frac{\partial \Lambda}{\partial \lambda_N}, [S_1, \Lambda], \dots, [S_{N(N-1)}, \Lambda]) \right| \\ &= \left| \det \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & \vdots & \ddots & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (S_1^R)_{12}(\lambda_2 - \lambda_1) & \cdots & \cdots & (S_{N(N-1)}^R)_{12}(\lambda_2 - \lambda_1) \\ 0 & \cdots & 0 & (S_1^I)_{12}(\lambda_2 - \lambda_1) & \cdots & \cdots & (S_{N(N-1)}^I)_{12}(\lambda_2 - \lambda_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & (S_1^R)_{N-1N}(\lambda_N - \lambda_{N-1}) & \cdots & \cdots & (S_{N(N-1)}^R)_{N-1N}(\lambda_N - \lambda_{N-1}) \end{pmatrix} \right| \\ &= |\det(A)| \prod_{i < j} |\lambda_j - \lambda_i|^2 \end{aligned}$$

where S_i^R and S_i^I the real and imaginary parts of S_i and A is the block matrix

$$A = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & S \end{pmatrix}.$$

Since A only depends on the variables p_i , write $|\det(A)| = g(p_1, \dots, p_{N(N-1)})$ and the proof is complete. \square

Note that since $\det(H) = \det(\Lambda) = \prod_{i=1}^N \lambda_i$ and $Tr(H) = Tr(\Lambda) = \sum_{i=1}^N \lambda_i$, it holds that $\omega_X^{(N,M)}(H) = \omega_X^{(N,M)}(\lambda_1, \dots, \lambda_N)$ where $\omega_X^{(N,M)}$ is defined in (3.3). Thus, both (3.1) and (3.2) can then be written as

$$d\mathbb{P}_X^{(N,M)}(\lambda_1, \dots, \lambda_N, p_1, \dots, p_{N(N-1)}) = \omega_X^{(N,M)}(\lambda) \det(A) \prod_{i < j} |\lambda_j - \lambda_i|^2 \prod_{i=1}^N d\lambda_i \prod_{i=1}^{N(N-1)} dp_i.$$

There is no interest in the variables p_i , so define

$$dP_X^{(N,M)}(\lambda_1, \dots, \lambda_N) = \underbrace{\int \cdots \int}_{N(N-1)} d\mathbb{P}_X^{(N,M)}(\lambda_1, \dots, \lambda_N, p_1, \dots, p_{N(N-1)}) dp_1 \cdots dp_{N(N-1)} \quad (3.6)$$

where the integrals are over the variables p_i . Writing out the special cases of (3.6) explicitly,

$$dP_G^{(N)}(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_{N,G}} \prod_{i=1}^N e^{-\frac{1}{2N}\lambda_i^2} \prod_{i < j} |\lambda_j - \lambda_i|^2 \prod_{i=1}^N d\lambda_i \quad (3.7)$$

$$dP_W^{(N,M)}(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_{N,M,W}} \prod_{i=1}^N \lambda_i^{N-M} e^{-\frac{\lambda_i}{2}} \prod_{i < j} |\lambda_j - \lambda_i|^2 \prod_{i=1}^N d\lambda_i. \quad (3.8)$$

3.1.2 Determinantal forms

GUE and WCE have the interesting property that they are determinantal processes, meaning their density can be written as a determinant of some kernel function. The following two propositions demonstrate this property.

Proposition 3.2 For (3.7),

$$P_G^{(N)}(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_{N,G}} \det[K_N^{(G)}(\lambda_i, \lambda_j)]_{1 \leq i, j \leq N} \quad (3.9)$$

where $K_N^{(G)}(\lambda_i, \lambda_j) = \sum_{k=1}^N \phi_{k-1}(\lambda_i) \phi_{k-1}(\lambda_j)$, $\phi_k(\lambda) = e^{-\frac{1}{4N}\lambda^2} \frac{H_k(\frac{\lambda}{\sqrt{2N}})}{\sqrt{2\pi N} 2^k (k!)}$ and $H_k(\lambda)$ are the Hermite polynomials.

Proof First, note that $\prod_{i < j} |\lambda_j - \lambda_i| = \det[\lambda_i^{j-1}]_{1 \leq i, j \leq N}$, which is the Vandermonde determinant. Now, adding appropriate multiples of the first column to the second, multiples of first and second columns to the third column etc. it follows that $\det[\lambda_i^{j-1}] = \det[\pi_{j-1}(\lambda_i)]$ where $\pi_j(x)$ are polynomials of the j 'th degree with the leading coefficient 1. Then

$$\begin{aligned} P_G^{(N)}(\lambda_1, \dots, \lambda_N) &= \frac{1}{Z_{N,G}} \prod_{i=1}^N e^{-\frac{1}{2N}\lambda_i^2} \prod_{i < j} |\lambda_j - \lambda_i|^2 \\ &= \frac{1}{Z_{N,G}} \prod_{i=1}^N e^{-\frac{1}{2N}\lambda_i^2} \det[\pi_{j-1}(\lambda_i)]^2 \\ &= \frac{1}{Z_{N,G}} \det[e^{-\frac{1}{4N}\lambda_i^2} \pi_{j-1}(\lambda_i)]^2 \\ &= \frac{1}{Z_{N,G}} \det[\phi_{j-1}(\lambda_i)]^2 \end{aligned}$$

where $\phi_{j-1}(\lambda_i) = e^{-\frac{1}{4N}\lambda_i^2} \pi_{j-1}(\lambda_i)$. The next natural step is to choose π_i so that they are orthonormal with respect to $e^{-\frac{1}{2N}\lambda_i^2}$. The orthogonality is required in lemma 3.4. Since they are generated by a similar weight, choose appropriately scaled Hermite polynomials as in the claim. Then

$$\begin{aligned} P_G^{(N)}(\lambda_1, \dots, \lambda_N) &= \frac{1}{Z_{N,G}} \det[\phi_{j-1}(\lambda_i)]^2 \\ &= \frac{1}{Z_{N,G}} \det\left[\sum_{k=1}^N \phi_{k-1}(\lambda_i) \phi_{k-1}(\lambda_j)\right] \\ &= \frac{1}{Z_{N,G}} \det[K_N^{(G)}(\lambda_i, \lambda_j)]_{1 \leq i, j \leq N} \end{aligned}$$

where $K_N^{(G)}(\lambda_i, \lambda_j) = \sum_{k=1}^N \phi_{k-1}(\lambda_i) \phi_{k-1}(\lambda_j)$ is the kernel of this determinantal point process. Equation (3.1) has then been reduced to a determinant of a simple kernel function. This proves the claim. \square

The determinantal form for WCE is very similar in both the claim and the proof.

Proposition 3.3 *For (3.8),*

$$P_W^{(N,M)}(\lambda_1, \dots, \lambda_N) = \frac{1}{Z_{N,M,W}} \det[K_{(N,M)}^W(\lambda_i, \lambda_j)]_{1 \leq i, j \leq N} \quad (3.10)$$

where $K_{(N,M)}^W(\lambda_i, \lambda_j) = \sum_{k=1}^N \phi_{k-1}(\lambda_i) \phi_{k-1}(\lambda_j)$, $\phi_k(\lambda) = \sqrt{\frac{k!}{2^{N-M+1} \Gamma(N-M+1+k)}} \lambda^{\frac{N-M}{2}} e^{-\frac{1}{4}\lambda} L_k^{(N-M)}(\frac{\lambda}{2})$ and $L_k^{(\alpha)}(\lambda)$ are the generalized Laguerre polynomials.

Proof Exactly as in the proof of proposition 3.2, first write the Vandermonde as a determinant of generalized Laguerre polynomials and absorb the other terms into the determinant.

$$\begin{aligned} P_W^{(N,M)}(\lambda_1, \dots, \lambda_N) &= \frac{1}{Z_{N,M,W}} \prod_{i < j} |\lambda_j - \lambda_i|^2 \prod_{i=1}^N \lambda_i^{N-M} e^{-\frac{\lambda_i}{2}} \\ &= \frac{1}{Z_{N,M,W}} \det[\lambda_i^{j-1}]^2 \prod_{i=1}^N \lambda_i^{N-M} e^{-\frac{\lambda_i}{2}} \\ &= \frac{1}{Z_{N,M,W}} \det[\lambda_i^{\frac{N-M}{2}} e^{-\frac{\lambda_i}{4}} L_{j-1}^{(N-M)}(\lambda_i)]^2 \\ &= \frac{1}{Z_{N,M,W}} \det\left[\sum_{k=1}^N \phi_{k-1}(\lambda_i) \phi_{k-1}(\lambda_j)\right] \\ &= \frac{1}{Z_{N,M,W}} \det[K_{(N,M)}^W(\lambda_i, \lambda_j)] \end{aligned}$$

This completes the proof. \square

3.1.3 n-point correlation functions

The density of eigenvalues can then be taken further by defining the n-point correlation functions for $n \leq N$. These are densities on subsets of all eigenvalues. They are obtained from $P_X^{(N,M)}$ by integration; The following lemma makes finding the correlation functions from equations (3.9) and (3.10) simple.

Lemma 3.4 *Let $A_N(x) = [A_{ij}]_{1 \leq i, j \leq N}$, $x \in \mathbb{R}^N$ be an $N \times N$ matrix such that*

(i) $A_{ij} = f(x_i, x_j)$ for some given measurable function $f : \mathbb{R}^2 \rightarrow \mathbb{C}$.

(ii) $\int f(x, y) f(y, z) d\mu(y) = f(x, z)$ for some measure $d\mu$ on \mathbb{R} .

Then

$$\int \det(A_N)_{1 \leq i, j \leq N} d\mu(x_N) = \det(A_N)_{1 \leq i, j \leq N-1} \left(\int f(x, x) d\mu(x) - N + 1 \right) \quad (3.11)$$

Proof This is lemma 5.27 of [10]. \square

Note that both kernels $K_N^{(G)}$ and $K_{N,M}^{(W)}$ satisfy the conditions of lemma 3.4 due to the normalization of the chosen orthogonal polynomials. The kernels also satisfy

$$\begin{aligned}\int_{-\infty}^{\infty} K_N^{(G)}(\lambda, \lambda) d\lambda &= \sum_{k=1}^N \int_{-\infty}^{\infty} \frac{e^{-\frac{\lambda^2}{2N}}}{\sqrt{2\pi n} 2^{k-1} (k-1)!} H_{k-1}\left(\frac{\lambda}{\sqrt{2n}}\right) H_{k-1}\left(\frac{\lambda}{\sqrt{2n}}\right) d\lambda = N \\ \int_0^{\infty} K_{N,M}^{(W)}(\lambda, \lambda) d\lambda &= \sum_{k=1}^N \int_0^{\infty} \frac{(k-1)!}{2^{N-M+1} \Gamma(N-M+k)} \lambda^{N-M} e^{-\frac{\lambda}{2}} L_{k-1}^{(N-M)}\left(\frac{\lambda}{2}\right) L_{k-1}^{(N-M)}\left(\frac{\lambda}{2}\right) d\lambda = N\end{aligned}$$

with respect to the Lebesgue measure. One can then define the n -point correlation function:

$$\begin{aligned}\rho^{(N)}(\lambda_1, \dots, \lambda_n) &= \frac{N!}{(N-n)!} \int \mathbb{P}^{(N)}(\lambda_1, \dots, \lambda_N) d\lambda_{n+1} \dots d\lambda_N \\ &= \frac{N!}{Z_N (N-n)!} \int \det[K_N(\lambda_i, \lambda_j)]_{1 \leq i, j \leq N} d\lambda_{n+1} \dots d\lambda_N \\ &= \frac{N!}{Z_N (N-n)!} \left(\int K_N(x, x) dx - N + 1 \right) \int \det[K_N(\lambda_i, \lambda_j)]_{1 \leq i, j \leq N-1} d\lambda_{n+1} \dots d\lambda_{N-1} \\ &= \frac{N!}{Z_N (N-n)!} \det[K_N(\lambda_i, \lambda_j)]_{1 \leq i, j \leq n} \prod_{k=n+1}^N (N-k+1) \\ &= \frac{N!}{Z_N} \det[K_N(\lambda_i, \lambda_j)]_{1 \leq i, j \leq n}\end{aligned}\tag{3.12}$$

From the condition $\int \mathbb{P}^{(N)}(\lambda_1, \dots, \lambda_N) d\lambda_1 \dots d\lambda_N = 1$ it follows that $Z_N = N!$. The final forms for the n -points functions for $n \leq N$ are

$$P^{(N)}(\lambda_1, \dots, \lambda_N) = \frac{1}{N!} \det[K_N(\lambda_i, \lambda_j)]_{1 \leq i, j \leq N}\tag{3.13}$$

and

$$\rho^{(N)}(\lambda_1, \dots, \lambda_n) = \det[K_n(\lambda_i, \lambda_j)]_{1 \leq i, j \leq n}\tag{3.14}$$

These will now be used to find the so called *gap probability*.

3.2 Largest Eigenvalue Statistics

Gap probability can be defined as the probability that no eigenvalue resides in a given Borel set. With the help of the next proposition, calculating it for the largest eigenvalue of the two ensembles will be trivial.

Proposition 3.5 *Consider a point process for which all correlation functions exist. Let ϕ be a complex-valued, bounded, measurable function with a bounded support. When the support of ϕ is contained in a bounded, measurable set B and*

$$\sum_{n=0}^N \frac{\|\phi\|_{\infty}^n}{n!} \int_{B^N} \rho^{(N)}(\lambda_1, \dots, \lambda_n) d^n \lambda < \infty$$

Then,

$$\mathbb{E}(\prod_{k=1}^N (1 + \phi(\lambda_k))) = \sum_{n=0}^N \frac{1}{n!} \int_{\Lambda^n} \prod_{k=1}^n \phi(\lambda_k) \rho^{(N)}(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n \quad (3.15)$$

Proof can be found in [11], prop. 2.2. \square

Let $\phi = \mathbb{1}$, that is, the indicator function on some interval of the real line. Then the hypothesis of the above proposition is obviously satisfied. It can then be seen that

$$\begin{aligned} \mathbb{P}_G^{(N)}(\lambda_{max} \leq t) &= \mathbb{P}_G^{(N)}(\bigcap_{k=1}^N (\lambda_k \leq t)) = \mathbb{E}_G(\prod_{k=1}^N (1 - \mathbf{1}_{(t, \infty)}(\lambda_k))) \\ &= \sum_{n=0}^N \frac{(-1)^n}{n!} \int_{(t, \infty)^n} \rho^{(N)}(\lambda_1, \dots, \lambda_n) d\lambda_1 \dots d\lambda_n \\ &= \sum_{n=0}^N \frac{(-1)^n}{n!} \int_{(t, \infty)^n} \det(K_N^G(\lambda_i, \lambda_j))_{1 \leq i, j \leq n} d\lambda_1 \dots d\lambda_n \end{aligned} \quad (3.16)$$

Similarly, $\mathbb{P}_W(\lambda_{max} \leq t)$ has the same kind of determinantal form with identical proof. On the other hand, directly integrating equation (3.8) yields a form for this probability more useful for comparison with later results. Thus, after some rescaling and a new normalization constant,

$$\begin{aligned} \mathbb{P}_W(\lambda_{max} \leq t) &= \frac{1}{Z_{N,M}} \int_{[0, t]^N} \prod_{j>i} |\lambda_j - \lambda_i|^2 \prod_{j=1}^N \lambda_i^{N-M} e^{-\lambda_i} d^N \lambda \\ &= \sum_{n=0}^N \frac{(-1)^n}{n!} \int_{(t, \infty)^n} \det(K_N^W(\lambda_i, \lambda_j))_{1 \leq i, j \leq n} d\lambda_1 \dots d\lambda_n \end{aligned} \quad (3.17)$$

where $Z_{N,M}$ is the limit of the integral as $t \rightarrow \infty$.

4 Simple Exclusion Process (SEP)

4.1 The model

In 1970, a seminal paper on interaction of Markov Processes [2] was published by F. Spitzer. This can be said to be the beginning of the asymmetric exclusion process. The simple exclusion process is a stochastic model for interacting particles on a lattice. A large number N of particles move randomly on a lattice with the property that two particles cannot occupy the same site at the same point in time. It is used to model things such as protein translation [12] and bears a close resemblance to the XXZ spin chain used in statistical physics. These two models are related through the equivalence of their respective infinitesimal generator and hamiltonian [13].

Let $\Lambda \subset \mathbb{Z}$ be the lattice. This can be either finite or infinite. The state space of one-dimensional SEP can be defined as $\Omega = \{0, 1\}^\Lambda$. Then $\omega(t) \in \Omega$ is a vector $\omega(t) = (\omega_x(t))_{x \in \Lambda}$, each $\omega_x(t)$ having values in $\{0, 1\} \forall t \in \mathbb{R}^+$. Now each $\omega(t)$ is a state of the system at time t with $\omega_x(t) = 1$ if there is a particle at $x \in \Lambda$ and $\omega_x(t) = 0$ if the site x is empty. Since there are at most $N = |\Lambda|$ particles, it is sometimes easier to consider the configuration space $X_t = \{x_1(t), \dots, x_N(t) \in \mathbb{Z} \mid x_1(t) < \dots < x_N(t) \forall t \in \mathbb{R}^+\}$

where $x_i(t)$ is the coordinate of the i 'th site from the left for which $\omega_x(t) = 1$. Denote the initial configuration by $Y = X_0 = \{y_1, \dots, y_N | y_1 < \dots < y_N\}$ and write $X = X_t$ when this is obvious.

Choose $\Lambda = \mathbb{Z}$, i.e. an infinite lattice.

The dynamics of the model is stochastic in the sense that each individual particle moves randomly when there are no particles nearby, and interacts with other nearby particles. On each particle a Poisson clock is attached; All particles' clocks are independent of each other and ring after an exponential waiting time with mean 1. When the clock of a particle rings, the particle attempts a jump to the right with probability p and left with the probability $q = 1 - p$. Simple refers to the fact that the jumps are always nearest neighbour. If the target site is occupied, the jump is suppressed.

The most natural way to start analysis of the model is to write down the evolution equation for the probability of a particular configuration. This is called the master equation and can be written as follows.

$$\frac{\partial P^t}{\partial t} = \mathbb{M}P^t \quad (4.1)$$

Here \mathbb{M} is the Markov operator and $P^t : X_t \mapsto [0, 1]$ the transition probability to configuration X in time t . Note that all contour integrals in this section should include the $\frac{1}{2\pi i}$ as a factor in front of the integral, but here, as in the future, dz should be read as $\frac{dz}{2\pi i}$ unless the factors are explicitly included.

4.2 Totally Asymmetric Simple Exclusion Process (TASEP)

The special case $q = 1$ ($p = 1$) is called the totally asymmetric simple exclusion process (TASEP). The particles can only move left (right), so the interactions for the m 'th particle can be reduced to two-body collisions with *only the next particle left of the m 'th one*, with indirect interaction with all the particles left of that particle. This holds for all particles, and so the distribution function for the leftmost particle should reduce to the case of having just a single free particle on Λ .

4.2.1 The Master Equation

For TASEP, equation (4.1) for the transition probability for particles moving to the left reads

$$\begin{aligned} \frac{d}{dt}P(x_1, \dots, x_N; t) &= \sum_{k=1}^N (P(x_1, \dots, x_k + 1, \dots, x_N; t) - P(x_1, \dots, x_N; t)) \\ &= \sum_{k=1}^N P(x_1, \dots, x_k + 1, \dots, x_N; t) - NP(x_1, \dots, x_N; t). \end{aligned} \quad (4.2)$$

This equation alone does not satisfy the properties of the model, but requires boundary conditions to include the actual exclusion property

$$P(x_1, \dots, x_k, x_{k+1} = x_k, \dots, x_N; t) = P(x_1, \dots, x_k, x_{k+1} = x_k + 1, \dots, x_N; t) \quad \forall t \geq 0, k = 1, \dots, N-1. \quad (4.3)$$

How this is derived is explained below. The transition probability must also satisfy the initial condition

$$P(X; 0) = \delta_{X,Y}. \quad (4.4)$$

Denote the transition probability with initial condition $P(X; 0) = \delta_{X,Y}$ by $P_Y(X; t)$.

The solution simultaneously satisfying equations (4.2), (4.3) and (4.4) can be found using a very powerful ansatz.

4.2.2 Bethe Ansatz

The name Bethe Ansatz originates from a paper by H. Bethe in 1931 [14] to obtain the eigensystem of the one-dimensional spin- $\frac{1}{2}$ XXX Heisenberg chain. It has since been used, surprisingly, to solve a wide range of specific models in quantum mechanics [15].

The exact form of transition probability for ASEP remained elusive for decades after the original paper by F. Spitzer [2]. In 1992, however, L. Gwa and H. Spohn proposed the use of Bethe Ansatz for ASEP [16]. The transition probability for TASEP was subsequently solved by Schutz in 1997 [17]. It was also proposed in the same paper that ASEP should be solvable in the same manner, and demonstrated the two particle solution. In their seminal paper from 2008, C. Tracy and H. Widom expanded on the solution of Schutz and obtained the solution for the transition probability of N-particle ASEP, along with the $N \rightarrow \infty$ limit [5].

Using the power of hindsight, the solution of the TASEP transition probability will be obtained using Bethe Ansatz in this section. For clarity, consider cases $N = 1$, $N = 2$ before attempting to find the complete solution.

One particle TASEP

Intuitively, one should expect a free particle solution with only left jumps. For one particle the master equation is

$$\frac{dP(x; t)}{dt} = P(x + 1; t) - P(x; t). \quad (4.5)$$

for all $x \in \Lambda$. Separate time and space coordinates by $P(x; t) = e^{-\epsilon t} P(x)$. Equation (4.5) becomes

$$\epsilon P(x) = -P(x + 1) + P(x)$$

which is solved by $P(x) = e^{ipx}$, $p \in [0, 2\pi)$. Now,

$$\begin{aligned} \epsilon e^{ipx} &= -e^{ip(x+1)} + e^{ipx} \\ \Rightarrow \epsilon_p &= 1 - e^{ip} \end{aligned} \quad (4.6)$$

and

$$P(x; t) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\epsilon_p t} e^{ipx} f(p) dp.$$

Using the initial condition (4.4), $f(p) = e^{ipy}$ (note that $f(p)$ is essentially the fourier transform of the y-centered delta function), and the solution now reads

$$\begin{aligned} P_Y(x; t) &= \frac{e^{-t}}{2\pi} \int_0^{2\pi} e^{e^{ip}t} e^{ip(x-y)} dp \\ &= \frac{e^{-t}}{2\pi i} \oint_{C_r} \frac{e^{tz}}{z^{y+1-x}} dz \\ &= \frac{t^{x-y}}{(x-y)!} e^{-t} \\ &= F_0(x-y; t) \end{aligned}$$

where C_r is any anticlockwise loop containing the origin and

$$F_n(x; t) = \oint_{C_r} e^{-\epsilon(z)t} \frac{(1-z)^n}{z^n} z^{x-1} dz \quad (4.7)$$

C_r is any anticlockwise loop with small enough radius to contain only the pole at the origin. F_n is chosen so that it works with the later solution for general n.

Two particle TASEP

The case of $N = 2$ is a bit more interesting; The boundary condition and the actual non-trivial Bethe ansatz have to be used for two particles. Recall that the particles move to the left ($q = 1$). The master equation reads

$$\frac{dP(x_1, x_2; t)}{dt} = P(x_1 + 1, x_2; t) + P(x_1, x_2 + 1; t) - 2P(x_1, x_2; t) \text{ for } x_2 - x_1 > 1 \quad (4.8)$$

and

$$\frac{dP(x_1, x_2; t)}{dt} = P(x_1, x_2 + 1; t) - P(x_1, x_2; t) \text{ for } x_2 - x_1 = 1 \quad (4.9)$$

with the boundary condition

$$P(x, x; t) = P(x - 1, x; t). \quad (4.10)$$

The boundary condition (4.10) can be found by demanding that equations (4.8) and (4.9) are simultaneously satisfied for each $t \in \mathbb{R}$ when $x_2 = x = x_1 + 1$.

Equation (4.9) contains the information that if the particles are located on neighbouring sites, the configuration can only be entered from the configuration $(x_1, x_2 + 1)$ and left from to only $(x_1 - 1, x_2)$. With the boundary condition, the range of Equation (4.8) is extended to include $x_2 - x_1 \geq 1$, since when $x_2 - x_1 = 1$, it follows from the boundary condition (4.10) that $P(x_1 + 1, x_2; t) - P(x_1, x_2; t) = 0$, leaving only equation (4.9). One can then proceed to solve the equation (4.8) by the familiar separation of time and space coordinates,

$$P(x_1, x_2; t) = e^{-\epsilon t} P(x_1, x_2). \quad (4.11)$$

Then equations (4.8) and (4.10) read

$$\begin{aligned} \epsilon P(x_1, x_2) &= 2P(x_1, x_2) - P(x_1 + 1, x_2) - P(x_1, x_2 + 1), \\ P(x, x) &= P(x - 1, x) \text{ for all } x \in \Lambda. \end{aligned} \quad (4.12)$$

Using Bethe's ansatz $P(x_1, x_2) = A_{12}e^{ip_1x_1+ip_2x_2} + A_{21}e^{ip_2x_1+ip_1x_2}$, it follows from (4.12) that

$$\epsilon_{p_1, p_2} = \epsilon_{p_1} + \epsilon_{p_2}$$

where ϵ_{p_i} are defined in equation (4.6). Using the boundary condition (4.10) one obtains

$$S_{12} := \frac{A_{12}}{A_{21}} = -\frac{e^{ip_1+ip_2} - e^{ip_1}}{e^{ip_1+ip_2} - e^{ip_2}} \text{ for } p_1, p_2 \in [0, 2\pi).$$

Integrating both sides of equation (4.11) over p_1 and p_2 , one obtains

$$\begin{aligned} \mathbb{P}_Y(x_1, x_2; t) &= \\ &= \int_0^{2\pi} \int_0^{2\pi} e^{-(\epsilon_{p_1} + \epsilon_{p_2})t} A_{12}(y_1, y_2) (e^{ip_1x_1+ip_2x_2} + S_{21}e^{ip_1x_2+ip_2x_1}) dp_1 dp_2 \\ &= \oint_{C_2} \oint_{C_1} e^{-\epsilon t} (z_1^{x_1-y_1-1} z_2^{x_2-y_2-1} - \frac{z_2}{z_1} \frac{1-z_1}{1-z_2} z_1^{x_2-y_1-1} z_2^{x_1-y_2-1}) dz_1 dz_2 \\ &= F_0(x_1 - y_1) F_0(x_2 - y_2) - F_1(x_2 - y_1) F_{-1}(x_1 - y_2) \\ &= \det[F_{i-j}(x_i - y_j)]_{1 \leq i, j \leq 2} \end{aligned}$$

A change of variables $z_i = e^{ip_i}$ was done in the second equality. All multiplicative constants are included in the measures dz_1 and dz_2 . The choice of $A_{12} = z_1^{-y_1} z_2^{-y_2}$ is motivated by the initial condition $\mathbb{P}_Y(X; 0) = \delta_{X, Y}$. The initial condition is satisfied for contours $C_1 = C_2 = \mathcal{C}_r$ which is a small origin-centered circle including only the pole at the origin.

N particle TASEP

What makes Bethe Ansatz superior in solving the N particle TASEP is that for more than two particles, no new kind of constraints are introduced. All configurations with two or more adjacent particles are reduced to just interaction of two particles at a time, making the generalization to an arbitrary number of particles fairly trivial. The master equation (4.1) for N particle TASEP reduces to form

$$P(x_1, \dots, x_N; t) = \sum_{n=1}^N P(x_1, \dots, x_n + 1, \dots, x_N; t) - NP(x_1, \dots, x_N; t) \quad (4.13)$$

for the case of no adjacent particles. If there are any particles with neighbours, the equation must satisfy

$$\begin{aligned} P(x_1, \dots, x_N; t) &= \sum_{n=1}^N P(x_1, \dots, x_n + 1, \dots, x_N; t) - NP(x_1, \dots, x_N; t) \\ &\quad - \sum_{x_i+1=x_{i+1}} (P(x_1, \dots, x_i + 1, \dots, x_N; t) - P(x_1, \dots, x_N; t)). \end{aligned} \quad (4.14)$$

The second term in the second sum comes from the fact that for each particle with another particle on the right, the configuration where the particle on the right has moved left is excluded. Since it is desirable to have an equation of the form (4.13) to solve, equating (4.13) and (4.14) separately for all possible configurations with adjacent particles at each time $t \in \mathbb{R}$ yields $N - 1$ boundary conditions of the form

$$P(\underbrace{x_1, \dots, x}_{i-1}, x, \dots, x_N; t) = P(\underbrace{x_1, \dots, x}_{i-1} - 1, x, \dots, x_N; t) \text{ for all } i \in \{1, \dots, N - 1\}, t \in \mathbb{R}. \quad (4.15)$$

Only two-particle interaction terms are independent, since terms with clusters of more than two particles can actually be reduced to the same constraints. Since there are N particles to solve for, adding the initial condition $P(X; 0) = \delta_{X,Y}$ gives N independent equations to solve, giving hope of finding a solution. Separating the time coordinate by

$$P(x_1, \dots, x_N; t) = e^{-\epsilon t} P(x_1, \dots, x_N) \quad (4.16)$$

and as before, using the ansatz

$$P(x_1, \dots, x_N) = \sum_{\sigma \in \mathbb{S}_N} A_{\sigma(1), \dots, \sigma(N)} e^{ip_{\sigma(1)}x_1 + \dots + ip_{\sigma(N)}x_N} \quad (4.17)$$

should give the solution. Substitute (4.16) into (4.13) to obtain

$$\epsilon \sum_{\sigma \in \mathbb{S}_N} A_{\sigma(1), \dots, \sigma(N)} \prod_{i=1}^N e^{ip_{\sigma(i)}x_i} = \sum_{\sigma \in \mathbb{S}_N} A_{\sigma(1), \dots, \sigma(N)} (N - \sum_{i=1}^N e^{ip_{\sigma(i)}}) \prod_{i=1}^N e^{ip_{\sigma(i)}x_i}. \quad (4.18)$$

The sum on the right-hand side factorizes giving

$$\epsilon = \sum_{i=1}^N (1 - e^{ip_i})$$

Then, substituting equation (4.17) in each of the boundary conditions (4.15) gives

$$A_{1,\dots,N} \sum_{\sigma \in \mathbb{S}_N} A_\sigma e^{ip_{\sigma(1)}x_1} \dots e^{ip_{\sigma(i)}x} e^{ip_{\sigma(i+1)}x} (1 - e^{-ip_{\sigma(i)}}) \dots e^{ip_{\sigma(N)}x_N} = 0 \quad \forall i \in \{1, \dots, N-1\}$$

where $A_\sigma = \frac{A_{\sigma(1), \dots, \sigma(N)}}{A_{1, \dots, N}}$. It turns out the solution is [5]

$$A_\sigma = \prod_{(\alpha, \beta) \text{ inversion in } \sigma} S_{\alpha\beta}$$

where $S_{\alpha\beta} = -\frac{e^{ip_\alpha + ip_\beta} - e^{ip_\beta}}{e^{ip_\alpha + ip_\beta} - e^{ip_\alpha}}$. An inversion in σ is an ordered pair $\{\sigma(i), \sigma(j)\}$ for which $i < j$ and $\sigma(i) > \sigma(j)$. The solution can again be obtained by integration and using $A_{1,\dots,N} = \prod_{i=1}^N e^{-ip_i y_i}$ with a change of variables $z_i = e^{ip_i}$. For a fixed σ , $1 - z_i$ occurs in A_σ $\sigma(i)$ times and $\frac{1}{1-z_i}$ i times. Thus, $A_\sigma = \text{sgn}(\sigma) \prod_{i=1}^N \frac{(1-z_i)^{\sigma(i)-i}}{z_i^{\sigma(i)-i}}$. The solution is

$$\begin{aligned} P(x_1, \dots, x_N; t | y_1, \dots, y_N; 0) &= \sum_{\sigma \in \mathbb{S}_N} \oint_{C_r} \dots \oint_{C_r} A_\sigma \prod_{i=1}^N (z_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{\epsilon(z_i)}) dz_1 \dots dz_N \\ &= \sum_{\sigma \in \mathbb{S}_N} \text{sgn}(\sigma) \prod_{i=1}^N \oint_{C_r} (1 - z_{\sigma(i)})^{\sigma(i)-i} z_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} e^{z_i t} e^{-t} dz_{\sigma(i)} \quad (4.19) \\ &= \det \left[\oint_{C_r} \frac{(1-z)^{j-i}}{z^{j-i}} z^{x_i - y_j - 1} e^{zt} e^{-t} dz \right]_{1 \leq i, j \leq N} \\ &= \det[F_{j-i}(x_i - y_j; t)]_{0 \leq i, j \leq N} \end{aligned}$$

which is a nice determinantal form for the transition probability. The proof that the transition probability is actually given by the right-hand side can be found by specializing theorem 4.3 to $q = 1$. The transition probability so obtained works as a starting point for several interesting theorems.

4.2.3 Connection to Random Matrix Theory

TASEP with particles moving to the right ($p = 1$) can be obtained in a very similar manner to the $q = 1$ case. Define a new F_n with

$$\begin{aligned} F_n(x; t) &= e^{-t} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{t^{k+x}}{(k+x)!} \\ &= \oint_{C_r} (1-z)^{-n} z^{x-1} e^{t(z^{-1}-1)} dz \end{aligned} \quad (4.20)$$

where C_r is the familiar counterclockwise contour around the origin with smaller radius than one. Then, for the case $p = 1$,

$$P_Y(X; t) = \det[F_{i-j}(x_i - y_j; t)]$$

as was shown by Schutz in [17]. In this section, assume $p = 1$.

There is a strong connection between the Totally Asymmetric Exclusion Process and Random Matrix Theory. It turns out the distribution of the largest eigenvalue in WCE as calculated in section 3 matches

the probability that the n 'th particle from the right ($n \leq N$) has jumped m times at time t with the step initial condition. This can be used to analyze the particle current at a specific site, and from that obtain macroscopic limits. This can be seen from the following proposition [18].

Proposition 4.1 *Denote by $x_n(t)$ the position of the n 'th particle from the right. Since the particles jump to the right, there is no need to account for the particles left of this particle. Choose step initial condition, i.e. $y_i = i - n$ and let $\mathbb{P}(n, m; t) = \mathbb{P}(x_n(t) = x_n(0) + m)$ be the probability that n 'th particle from the right has jumped m times at time t . Then,*

$$\mathbb{P}(n, m) = \prod_{k=1}^n \frac{1}{k!(m-k)!} \int_{[0,t]^n} \prod_{k=1}^n x_k^{n-m} e^{-x_k} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 d^n x. \quad (4.21)$$

Proof For later reference, some important identities of the kernel (4.20) will have to be stated.

Lemma 4.2 *The following properties hold for the kernel (4.20)*

- (i) $F_{n+1}(x; t) = \sum_{y \geq x} F_n(y; t)$.
- (ii) $\frac{d}{dt} F_n(x; t) = F_n(x-1; t) - F_n(x; t) = F_{n-1}(x-1; t)$.
- (iii) $F_n(x; t) = e^{-t} \sum_{k=0}^{|n|} (-1)^k \binom{|n|}{k} \frac{t^{k+x}}{(k+x)!}$ for $n \leq 0$.

Proof (i):

$$\begin{aligned} F_{n+1}(x; t) &= e^{-t} \sum_{k=0}^{\infty} \binom{k+n}{n} \frac{t^{k+x}}{(k+x)!} = e^{-t} \sum_{k=0}^{\infty} \frac{k+n}{n} \binom{k+n-1}{n-1} \frac{t^{k+x}}{(k+x)!} \\ &= F_n(x; t) + e^{-t} \sum_{k=0}^{\infty} \binom{k+n-1}{k-1} \frac{t^{k+x}}{(k+x)!} \\ &= F_n(x; t) + e^{-t} \sum_{k=0}^{\infty} \binom{k+n}{n} \frac{t^{k+n+1}}{(k+n+1)!} = F_n(x; t) + F_{n+1}(x+1; t) \\ &\Rightarrow F_{n+1}(x; t) = \sum_{y \geq x} F_n(y; t). \end{aligned}$$

In the fourth equality the fact $\binom{n+k-1}{k-1} = 0$ for $k \leq 0$ was used. (ii) can be proved with the help of the fifth equality in the last proof:

$$\begin{aligned} \frac{d}{dt} F_n(x; t) &= e^{-t} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{t^{k+x-1}}{(k+n-1)!} \\ &\quad - e^{-t} \sum_{k=0}^{\infty} \binom{k+n-1}{n-1} \frac{t^{k+x}}{(k+n)!} \\ &= F_n(x-1; t) - F_n(x; t) \\ &= F_{n-1}(x-1; t). \end{aligned}$$

(iii):

$$\begin{aligned}
F_n(x; t) &= e^{-t} \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{\Gamma(k+1)\Gamma(n)} \frac{t^{k+x}}{(k+x)!} \\
&= e^{-t} \left(\frac{t^x}{x!} + n \frac{t^{x+1}}{(x+1)!} + \frac{n(n+1)}{2!} \frac{t^{x+1}}{(x+2)!} + \frac{n(n+1)(n+2)}{3!} \frac{t^{x+3}}{(x+3)!} + \dots \right) \\
&= e^{-t} \sum_{k=0}^{|n|} (-1)^k \binom{|n|}{k} \frac{t^{k+x}}{(k+x)!}
\end{aligned}$$

and the proof is complete. \square

Write

$$\mathbb{P}(n, m; t) = \sum_{x_n=m}^{\infty} \sum_{x_{n-1}=m-1}^{x_n-1} \cdots \sum_{x_2=m-n+2}^{x_3-1} \sum_{x_1=m-n+1}^{x_2-1} \det[F_{i-j}(x_i - i + n)] \quad (4.22)$$

for the probability. First, summing over x_1 gives

$$\begin{aligned}
\mathbb{P}(n, m; t) &= \sum_{x_n=-m+n-1}^{-\infty} \sum_{x_{n-1}=-m+n-2}^{x_n-1} \cdots \sum_{x_2=-m+1}^{x_3-1} \\
&\det \left| \begin{pmatrix} F_1(m; t) - F_1(x_2 + n - 1; t) & \cdots & F_{-n+2}(m - n + 1; t) - F_{-n+2}(x_2; t) \\ F_1(x_2 - 1 + n; t) & \cdots & F_{-n+2}(x_2; t) \\ \vdots & \ddots & \vdots \\ F_{n-1}(x_n - 1 + n; t) & \cdots & F_0(x_n; t) \end{pmatrix} \right|
\end{aligned}$$

Then, adding the second row to the first one and repeating this for all the sums except the one over x_n . The x_n sum is just application of lemma 4.2, (i). The probability is then

$$\begin{aligned}
\mathbb{P}(n, m; t) &= \\
&\det \left| \begin{pmatrix} F_1(m; t) & F_0(m-1; t) & \cdots & F_{-n+2}(m-n+1; t) \\ F_2(m+1; t) & F_1(m; t) & \cdots & F_{-n+3}(m-n+2; t) \\ \vdots & \vdots & \ddots & \vdots \\ F_n(m+n-1; t) & F_{n-1}(m+n-2; t) & \cdots & F_1(m; t) \end{pmatrix} \right|.
\end{aligned}$$

Using lemma 4.2, (ii) to write the elements as integrals,

$$\begin{aligned}
\mathbb{P}(n, m; t) &= \\
&\det \left| \begin{pmatrix} \int_0^t F_0(m-1; t) dt & \int_0^t F_{-1}(m-2; t) dt & \cdots & \int_0^t F_{-n+1}(m-n; t) dt \\ \int_0^t F_1(m; t) dt & \int_0^t F_0(m-1; t) dt & \cdots & \int_0^t F_{-n+2}(m-n+1; t) dt \\ \vdots & \vdots & \ddots & \vdots \\ \int_0^t F_{n-1}(m+n-2; t) dt & \int_0^t F_{n-2}(m+n-3; t) dt & \cdots & \int_0^t F_0(m-1; t) dt \end{pmatrix} \right|.
\end{aligned}$$

Note that the first row is $F_{2-j}(m+1-j; t)$. The second row is $\int_0^t F_{2-j}(m-j+1; t)dt$. After partial integration this becomes $tF_{2-j}(m-j+1; t) - \int_0^t tF_{1-j}(m-j; t)dt$ after using lemma 4.2, (ii) for the derivative. Subtract t times the first row from the second row, yielding $-\int_0^t tF_{1-j}(m-j; t)dt$ for the second row. This can be repeated for every row, each time adding one more partial integration. The end result is

$$\begin{aligned} \mathbb{P}(n, m; t) &= \det \left| \begin{pmatrix} \int_0^t F_0(m-1; t)dt & \int_0^t F_{-1}(m-2; t)dt & \cdots & \int_0^t F_{-n+1}(m-n; t)dt \\ -\int_0^t tF_0(m-1; t)dt & -\int_0^t tF_{-1}(m-2; t)dt & \cdots & -\int_0^t tF_{-n+1}(m-n; t)dt \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{n+1}}{(n-1)!} \int_0^t t^{n-1}F_0(m-1; t)dt & \frac{(-1)^{n+1}}{(n-1)!} \int_0^t t^{n-1}F_{-1}(m-2; t)dt & \cdots & \frac{(-1)^{n+1}}{(n-1)!} \int_0^t t^{n-1}F_{-n+1}(m-n; t)dt \end{pmatrix} \right| \\ &= (-1)^{[n/2]} \prod_{j=1}^{n-1} \frac{1}{j!} \int_{[0, t]^n} t_2 t_3^2 \cdots t_n^{n-1} \times \det \left| \begin{pmatrix} F_0(m-1; t_1) & F_{-1}(m-2; t_1) & \cdots & F_{-n+1}(m-n; t_1) \\ F_0(m-1; t_2) & F_{-1}(m-2; t_2) & \cdots & F_{-n+1}(m-n; t_2) \\ \vdots & \vdots & \ddots & \vdots \\ F_0(m-1; t_n) & F_{-1}(m-2; t_n) & \cdots & F_{-n+1}(m-n; t_n) \end{pmatrix} \right| d^n t. \end{aligned}$$

Using lemma 4.2 (ii) and (iii) brings this to the form

$$\begin{aligned} \mathbb{P}(n, m; t) &= (-1)^{[n/2]} \prod_{j=1}^{n-1} \frac{1}{j!} \prod_{j=1}^n \frac{1}{(m-j)!} \int_{[0, t]^n} \prod_{j=1}^n (t_i^{n-m} e^{-t_i}) t_2 \cdots t_n^{n-1} \\ &\quad \times \det \left| \begin{pmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1^0 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2^0 \\ \vdots & \vdots & \ddots & \vdots \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n^0 \end{pmatrix} \right| d^n t. \end{aligned}$$

The integrand aside from the product of t_i^{i-1} and the determinant is symmetric. The determinant is antisymmetric. Thus, the integral stays unchanged on antisymmetrization of the product

$$\frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \text{sgn}(\sigma) \prod_{j=1}^n \tau_{\sigma(j)}^{j-1} = \det[t_i^{j-1}].$$

Switching all the rows in this to the same order as in the determinant already in the integrand, one obtains the Vandermonde determinant with the sign $(-1)^{[n/2]}$. This proves the claim. \square

4.3 Asymmetric Simple Exclusion Process (ASEP)

The case of $p \neq q \neq 1$ is called the asymmetric simple exclusion process (ASEP). The particle current has a bias in one direction, and a special case of this is the TASEP discussed in the previous subsection.

In this case, the procedure is very similar to TASEP: apply Bethe Ansatz to find the transition probabilities, write the cumulative distribution function for an arbitrary particle with $N \rightarrow \infty$ as a Fredholm determinant and apply asymptotic analysis on this. We will only consider asymptotics for step initial condition \mathbb{Z}^+ with a drift to the left ($q > p$) as originally done in [5]. Denote by $P_Y(x_1, \dots, x_N; t)$ the transition probability from configuration Y to X ; That is, the probability of being in a configuration $X = \{x_1, \dots, x_N\}$ at time t when starting from the configuration $Y = \{y_1, \dots, y_N\}$.

4.3.1 Transition probability

Let $p : \mathbb{Z}^N \times [0, \infty)$ be a function that satisfies the master equation for N -particle ASEP, that is

$$\begin{aligned} \frac{d}{dt} p(x_1, \dots, x_N; t) = & \sum_{k=1}^N (pp(x_1, \dots, x_{k-1}, x_k - 1, x_{k+1}, \dots, x_N; t) \\ & + qp(x_1, \dots, x_{k-1}, x_k + 1, x_{k+1}, \dots, x_N; t) - p(x_1, \dots, x_N)). \end{aligned} \quad (4.23)$$

Impose on p the initial condition

$$p(X; 0) = \delta_{X,Y} \text{ for } x_1 < \dots < x_N. \quad (4.24)$$

Finding the right boundary conditions is straightforward. Assume $x_1 < \dots < x_i = x_{i+1} - 1 < \dots < x_N$ for some $i \in \{1, \dots, N-1\}$. Then, from the exclusion condition, it follows that

$$\begin{aligned} \frac{d}{dt} p(x_1, \dots, x_N; t) = & \sum_{k=1}^N (pp(x_1, \dots, x_{k-1}, x_k - 1, x_{k+1}, \dots, x_N; t) \\ & + qp(x_1, \dots, x_{k-1}, x_k + 1, x_{k+1}, \dots, x_N; t) - p(x_1, \dots, x_N)) \\ & - pp(x_1, \dots, x_i, x_{i+1} - 1, x_{i+2}, \dots, x_N; t) \\ & - qp(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_N; t) + p(x_1, \dots, x_N; t). \end{aligned} \quad (4.25)$$

Obviously equations (4.23) and (4.25) both need to be satisfied, so set $x_i + 1 = x_{i+1}$ and require both equations are satisfied simultaneously at each point in time $t \in \mathbb{R}$, leading to the boundary conditions

$$p(x_1, \dots, x_i, x_i + 1, \dots, x_N; t) = pp(x_1, \dots, x_i, x_i, \dots, x_N; t) + qp(x_1, \dots, x_i + 1, x_i + 1, \dots, x_N; t) \quad (4.26)$$

for $i = 1, \dots, N$. For p satisfying equations, (4.23), (4.24) and (4.26) it holds that $p(X; t) = P_Y(X; t)$, $x_1 < \dots < x_N$. We are now ready to state the theorem determining P_Y .

Theorem 4.3 *The ASEP transition probability can be written*

$$P_Y(X; t) = \sum_{\sigma \in \mathbb{S}_N} \oint_{\mathcal{C}_r} \dots \oint_{\mathcal{C}_r} e^{\epsilon t} A_\sigma \prod_{k=1}^N z_k^{x_{\sigma(k)} - y_k - 1} d^N z \quad (4.27)$$

where \mathcal{C}_r are circular contours centered on the origin with radii small enough to have all poles of A_σ outside the contour, $A_\sigma = \prod_{(\alpha, \beta) \text{ inversion in } \sigma} S_{\alpha\beta}$, $S_{\alpha\beta} = -\frac{p+qz_\alpha z_\beta - z_\alpha}{p+qz_\alpha z_\beta - z_\beta}$ and $\epsilon = \sum_{k=1}^N (pz_k^{-1} + qz_k - 1)$

Proof Denote by $u_Y(X; t)$ the right-hand side of equation (4.27). As in [5], we will divide the proof in three parts as follows.

1. u_Y satisfies the master equation (4.23)
2. u_Y satisfies the boundary conditions (4.26)
3. u_Y satisfies the initial condition (4.24)

As mentioned earlier, this shows that $u_Y(X; t) = P_Y(X; t)$.

Proof of 1

This can be shown with a simple calculation.

$$\begin{aligned}
\frac{d}{dt}u_Y(X; t) &= \epsilon u_Y(X; t) \\
&= \sum_{i=1}^N \sum_{\sigma \in \mathbb{S}_N} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} e^{\epsilon t} A_\sigma z_1^{x_{\sigma(1)} - y_1 - 1} \cdots (pz_i^{-1} + qz_i - 1) z_i^{x_{\sigma(i)} - y_i - 1} \cdots z_N^{x_{\sigma(N)} - y_N - 1} d^N z \\
&= \sum_{i=1}^N (pu_Y(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_N; t) \\
&\quad + qu_Y(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_N; t) - u_Y(x_1, \dots, x_N; t))
\end{aligned} \tag{4.28}$$

which is exactly equation (4.23). \square

Proof of 2

The boundary condition (4.26) for $u_Y(X; t)$ can be manipulated, for all $i = 1, \dots, N - 1$, as follows

$$\begin{aligned}
&u_Y(x_1, \dots, x_i, x_i + 1, \dots, x_N; t) - pu_Y(x_1, \dots, x_i, x_i, \dots, x_N; t) \\
&\quad - qu_Y(x_1, \dots, x_i + 1, x_i + 1, \dots, x_N; t) = 0 \\
&\Leftrightarrow \sum_{\sigma \in \mathbb{S}_N} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} e^{\epsilon t} A_\sigma (z_{\sigma(i+1)} - p - qz_{\sigma(i)}z_{\sigma(i+1)}) \prod_{j=1}^N z_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} d^N z = 0 \\
&\Leftarrow - \sum_{\sigma \in \mathbb{S}_N} A_\sigma (p + qz_{\sigma(i)}z_{\sigma(i+1)} - z_{\sigma(i+1)}) \prod_{j=1}^N z_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} = 0 \\
&= \sum_{\sigma \in \mathbb{S}_N} A_\sigma S_{\sigma(i+1)\sigma(i)} (p + qz_{\sigma(i)}z_{\sigma(i+1)} - z_{\sigma(i)}) \prod_{j=1}^N z_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1}.
\end{aligned} \tag{4.29}$$

The result is that showing that the last equality holds is sufficient to prove the claim. The last factor is a product over all $z_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1}$ where $x_i = x_{i+1}$, so that the factor remains unchanged when exchanging $\sigma(i)$ with $\sigma(i+1)$. The value of the whole sum with the two entries interchanged does not change, and reads

$$- \sum_{\sigma \in \mathbb{S}_N} A_{\sigma'} (p + qz_{\sigma(i)}z_{\sigma(i+1)} - z_{\sigma(i)}) \prod_{j=1}^N z_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} = 0. \tag{4.30}$$

where $\sigma'(i) = \sigma(i+1)$ and $\sigma'(i+1) = \sigma(i)$. Summing equations (4.29) and (4.30) gives the condition

$$A_\sigma S_{\sigma(i+1)\sigma(i)} = A_{\sigma'} \text{ for each } i = 1, \dots, N - 1. \tag{4.31}$$

Thus, the whole proof reduces to showing this holds. The following part of the proof was done by C. Tracy and H. Widom in [5]. Suppose $\sigma(i) > \sigma(i+1)$. Then $\{\sigma(i), \sigma(i+1)\}$ is an inversion in σ but not in σ' . Thus $S_{\sigma(i)\sigma(i+1)}$ is a factor in A_σ but not in $A_{\sigma'}$ with all the other factors the same. $S_{\alpha\beta}S_{\beta\alpha} = 1$ holds for $\alpha, \beta \in \{1, \dots, N\}$, so

$$A_{\sigma'} S_{\sigma(i)\sigma(i+1)} = A_\sigma \Leftrightarrow A_{\sigma'} = A_\sigma S_{\sigma(i+1)\sigma(i)}.$$

This proves the claim. \square

Proof of 3

The initial condition $u_Y(X; 0) = \delta_{X,Y}$ is satisfied by the term $\sigma = id$ on the right-hand side of equation (4.27) as follows. For $\sigma = id$, the summand equals

$$\oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} A_{id} \prod_{j=1}^N z_j^{x_j - y_j - 1} d^N z.$$

Note that $A_{id} = 1$. If $X = Y$, $x_j = y_j$ for all $j = 1, \dots, N$, and so the integral becomes

$$\left(\oint_{\mathcal{C}_r} z^{-1} dz \right)^N = 1.$$

For $X \neq Y$, either $x_j < y_j$ or $x_j > y_j$. In either case, the integral vanishes. Thus it is sufficient to show that

$$\sum_{\sigma \neq id} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} A_\sigma \prod_{j=1}^N z_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} d^N z = 0$$

for $x_1 < \dots < x_N$. For future convenience, write the above equation as $\sum_{\sigma \neq id} I(\sigma)$. Choose some integer $n \in [1, N)$. Then, fix $n-1$ numbers $i_1, \dots, i_{n-1} \in [1, N)$ and define

$$A = \{i_1, \dots, i_{n-1}\}.$$

For each A, further define

$$\mathbb{S}_N(A) = \{\sigma \in \mathbb{S}_N \mid \sigma(1) = i_1, \dots, \sigma(n-1) = i_{n-1}, \sigma(n) = N\}.$$

and $B = (A \cup \{N\})^c$.

Lemma 4.4 *For each A, it holds that*

$$\sum_{\sigma \in \mathbb{S}_N(A)} I(\sigma) = 0.$$

Proof Note that for each term in the sum, $\sigma \in \mathbb{S}_N(A)$, the only inversions involving N are (N, i) where $i \in B$. Thus

$$I(\sigma) = \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} \prod_{i \in B} S_{Ni} \prod_{j=1}^N z_j^{x_{\sigma^{-1}(j)} - y_j - 1} \prod_{\substack{N > \beta > \alpha \\ \sigma^{-1}(\beta) < \sigma^{-1}(\alpha)}} S_{\beta\alpha} d^N z.$$

Make a change of variables $z_N = \frac{\eta}{\prod_{j < N} z_j}$ so that the η integral is over a contour $\mathcal{C}_{r,N}$. The integral becomes

$$I(\sigma) = \oint_{\mathcal{C}_{r,N}} \oint_{\mathcal{C}_r} \dots \oint_{\mathcal{C}_r} (-1)^{|B|} \prod_{i \in B} \frac{p + q\eta \prod_{j \neq i, N} z_j^{-1} - \eta \prod_{j=1}^{N-1} z_j^{-1}}{p + q\eta \prod_{j \neq i, N} z_j^{-1} - z_i} \eta^{x_n - y_N - 1} \prod_{j=1}^{N-1} z_j^{x_{\sigma^{-1}(j)} - x_n + y_N - y_j - 1} \\ \times \prod_{\substack{N > \beta > \alpha \\ \sigma^{-1}(\beta) < \sigma^{-1}(\alpha)}} S_{\beta\alpha} d^{N-1} z d\eta. \quad (4.32)$$

Since $1 \leq n < N$, find out what happens to the summand $I(\sigma)$ for different n . First,

Lemma 4.5 *For $n = N - 1$, $I(\sigma) = 0$.*

Proof Since $n = N - 1$, $|B| = 1$. The product over $i \in B$ in equation (4.32) is analytic inside the z_i contour, except for a simple pole at the origin. Similarly, the power of z_i in the next factor is $z_i^{x_N - x_{N-1} + y_N - y_i - 1}$. As $x_N > x_i$ and $y_N > y_i$ for all $i = 1, \dots, N - 1$, the exponent is strictly positive. Thus the integrand is analytic inside the z_i contour, implying that the integral vanishes. \square

Lemma 4.6 *When $n < N - 1$, all $I(\sigma)$ where $\sigma \in \mathbb{S}_N(A)$ are sums over lower order integrals, in each of which the product over $i \in B$ is replaced by a factor depending on A , leaving the other factors unchanged. In each integral, some z_i with $i \in B$ is equal to some other z_j with $j \in B$.*

Proof First, alter the contours slightly so that $z_i \in \mathcal{C}_{r_i}$ where the r_i are all different. For example, scale $r_i = (1 + i\epsilon)r_i$ for some small enough $\epsilon > 0$. The goal is then to shrink some of the contours \mathcal{C}_{r_i} for which $i \in B$. Since r_i are all small enough to not include any poles of $S_{\beta\alpha}$ away from the origin, the only poles that can be passed when shrinking a contour come from the $\prod_{i \in B}$ factor of the integrand.

Take $j = \max(B)$. Now see what happens when shrinking the z_j -contour. The integrand is analytic at $z_j = 0$ since there is a simple pole at $z_j = 0$ and the power of z_j is, as above, positive. Choose $k \in B$ so that $k \neq j$. During the deformation of the z_j contour, a pole at

$$z_j = \frac{q\eta \prod_{i \neq j, N, k} z_i}{z_k - p}$$

is passed. Then shrink the z_k contour. Now, instead, there is a pole of order two at $z_k = 0$. However, the exponent of z_k is greater than 1, and as such, the integrand is still analytic at $z_k = 0$. If $r_i < r_k$ for $i \neq j, k$, there is also a pole at $z_k = z_i$. Shrinking the z_j contour, and the z_k contours for $k \neq j$ after that, a sum of lower order integrals with two z_i , $i \in B$, being the same is obtained. This proves the claim. \square

Lemma 4.7 *For each $I(\sigma)$ in lemma 4.6, there is a partition of $\mathbb{S}_N(A)$ into pairs σ, σ' such that $I(\sigma) + I(\sigma') = 0$ for each pair.*

Proof First, look at an integral with $z_i = z_j$. Pair σ and σ' if $\sigma^{-1}(i) = \sigma'^{-1}(j)$, $\sigma^{-1}(j) = \sigma'^{-1}(i)$ and $\sigma^{-1}(k) = \sigma'^{-1}(k)$ for $k \neq i, j$. In equation (4.32), the product of z_j remains the same for σ and σ' for $z_i = z_j$. Assuming $i < j$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$ implies $\sigma'^{-1}(j) < \sigma'^{-1}(i)$ and thus $S_{j,i}$ appears only for

σ' but not σ . Since $z_i = z_j$, $S_{j,i} = -1$.

It is then sufficient to show that for any $k \neq i, j$, the product of the S-factors remains unchanged for σ, σ' when $z_i = z_j$. As noted by Tracy-Widom in [5], there are nine different cases. The following listing includes all these cases. The first column is the position of k with respect to i and j , the second is similarly for the position of $\sigma^{-1}(k)$, the third column is the product of S-factors for σ and the fourth column is the product of S-factors for σ' .

$k < i < j$	$\sigma^{-1}(k) < \sigma^{-1}(i) < \sigma^{-1}(j)$	1	S_{ji}
$k < i < j$	$\sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)$	S_{ik}	$S_{ji}S_{jk}$
$k < i < j$	$\sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k)$	$S_{jk}S_{ik}$	$S_{ji}S_{jk}S_{ik}$
$i < j < k$	$\sigma^{-1}(k) < \sigma^{-1}(i) < \sigma^{-1}(j)$	$S_{ki}S_{kj}$	$S_{ji}S_{ki}S_{kj}$
$i < j < k$	$\sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)$	S_{kj}	$S_{ji}S_{ki}$
$i < j < k$	$\sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k)$	1	S_{ji}
$i < k < j$	$\sigma^{-1}(k) < \sigma^{-1}(i) < \sigma^{-1}(j)$	S_{ki}	$S_{ji}S_{ki}$
$i < k < j$	$\sigma^{-1}(i) < \sigma^{-1}(k) < \sigma^{-1}(j)$	1	$S_{ki}S_{jk}S_{ji}$
$i < k < j$	$\sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k)$	S_{jk}	$S_{jk}S_{ji}$

Note that $z_i = z_j$, $S_{j,i} = 1$. Using $S_{\beta\alpha}S_{\alpha\beta} = 1$, one can see that all the cases are the same for σ and σ' . This finishes the proof. \square

To prove the claim of lemma 4.4, note that for $n = N - 1$, lemma 4.5 gives the result. For $n < N - 1$, lemmas 4.6 and 4.7 give the result. This proves the claim. \square

To show that $\sum_{\sigma \neq id} I(\sigma) = 0$, use induction. For $N = 2$, the result directly follows from lemma 4.5.

Assume then that the result holds for $N-1$. If $\sigma(N) = N$, there are no S-factors involving N , and thus the result follows directly from the induction hypothesis. In the case $\sigma(N) < N$, the set of such permutations is a disjoint union of $\mathbb{S}_N(A)$ for different A . Then the result follows directly from lemma 4.4 since the sum over a disjoint union equals the sum of sums over the disjoint sets. \square

4.3.2 One-point functions

Denote by $\mathbb{P}(x_m(t) = x)$ the one-point function of ASEP, ie. the probability that the m 'th particle from left occupies site x at time t . It will be instructive to first consider the leftmost particle. Following [5], define

$$I(x, Y, z) = \prod_{i < j} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{1 - \prod_{k=1}^N z_k}{\prod_{k=1}^N (1 - z_k)} e^{\epsilon t} \prod_{k=1}^N z_k^{x - y_k - 1} \quad (4.33)$$

$$I_f(z) = \prod_{i < j} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{f(z_1, \dots, z_N)}{\prod_{k=1}^N (1 - z_k)} \quad (4.34)$$

and, for a subset $S = \{s_1, \dots, s_n\} \subset \{1, \dots, N\}$,

$$I_{f,S}(z) = \prod_{\substack{i < j \\ i, j \in S}} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{f(z_{s_1}, \dots, z_{s_n})}{\prod_{k \in S} (1 - z_k)} \quad (4.35)$$

where $f(z_{s_1}, \dots, z_{s_n})$ is the function f with all $z_i = 1$ for $i \notin S$. Also introduce the notation

$$\begin{aligned} [N] &= \frac{p^N - q^N}{p - q} \\ [N]! &= [N][N-1] \dots [1] \\ \begin{bmatrix} N \\ m \end{bmatrix} &= \frac{[N]!}{[N-m]![m]!} \end{aligned}$$

State the following two identities for use in most of the later proofs.

Lemma 4.8

$$\sum_{\sigma \in \mathbb{S}_N} A_\sigma \prod_{i=2}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} = \frac{p^{\frac{N(N-1)}{2}}}{\prod_{i < j} (p - qz_i z_j - z_i)} \frac{\prod_{i < j} (z_j - z_i)}{\prod_{j=1}^N (1 - z_j)} \left(1 - \prod_{j=1}^N z_j\right) \quad (4.36)$$

Proof Note that

$$\begin{aligned} A_\sigma &= (-1)^{inv(\sigma)} \prod_{j=2}^N \prod_{\substack{i=1 \\ \sigma(j) < \sigma(i)}}^{j-1} \frac{p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}}{p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(j)}} \\ &= sgn(\sigma) \prod_{j=2}^N \prod_{i=1}^{j-1} \frac{p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}}{p + qz_i z_j - z_i} \end{aligned}$$

which gives the denominator in the claim. Denote by A_σ^* the A-factor without the denominator. Then

$$\begin{aligned}
\sum_{\sigma \in \mathbb{S}_N} A_\sigma^* \prod_{i=2}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} &= \sum_{k=1}^N \sum_{\sigma \in \mathbb{S}_N, \sigma(1)=k} \operatorname{sgn}(\sigma) \prod_{i < j} (p - qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}) \prod_{i=2}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} \\
&= \sum_{k=1}^N \sum_{\substack{\sigma \in \mathbb{S}_N \\ \sigma(1)=k}} \operatorname{sgn}(\sigma) \left(\prod_{j=2}^N (p - qz_k z_{\sigma(j)} - z_k) \frac{\prod_{j=2}^N z_{\sigma(j)}}{1 - \prod_{j=2}^N z_{\sigma(j)}} \right) \\
&\times \prod_{i=2}^{N-1} \prod_{j=i+1}^N (p - qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}) \prod_{i=3}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} \\
&= \sum_{k=1}^N \left(\prod_{j \neq k} (p - qz_j z_k - z_k) \frac{\prod_{j \neq k} z_j}{1 - \prod_{j \neq k} z_j} \right) \sum_{\substack{\sigma \in \mathbb{S}_N \\ \sigma(1)=k}} \operatorname{sgn}(\sigma) \prod_{i=2}^{N-1} \prod_{j=i+1}^N (p - qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}) \prod_{i=3}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} \\
&= \sum_{k=1}^N \left(\prod_{j \neq k} (p - qz_j z_k - z_k) \frac{\prod_{j \neq k} z_j}{1 - \prod_{j \neq k} z_j} \right) \sum_{\substack{\sigma \in \mathbb{S}_N \\ \sigma(1)=k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{N-2} \prod_{j=i+1}^{N-1} (p - qz_{\sigma(i+1)}z_{\sigma(j+1)} - z_{\sigma(i+1)}) \\
&\times \prod_{i=2}^{N-1} \frac{\prod_{j=i}^{N-1} z_{\sigma(j+1)}}{1 - \prod_{j=i}^{N-1} z_{\sigma(j+1)}}.
\end{aligned}$$

Now, make a change of variables from σ to π by setting $\sigma(i+1) = \pi(i)$, $\sigma(1) = \pi(N)$, that is, π is σ cyclically shifted to the left by one. This can be achieved via $N-1$ transpositions, so $\operatorname{sgn}(\sigma) = (-1)^{N-1} \operatorname{sgn}(\pi)$. Now

$$\begin{aligned}
\sum_{\sigma \in \mathbb{S}_N} A_\sigma^* \prod_{i=2}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} &= \sum_{k=1}^N (-1)^{N-1} \left(\prod_{j \neq k} (p - qz_j z_k - z_k) \frac{\prod_{j \neq k} z_j}{1 - \prod_{j \neq k} z_j} \right) \\
&\times \sum_{\substack{\pi \in \mathbb{S}_N \\ \pi(N)=k}} \operatorname{sgn}(\pi) \prod_{i < j}^{N-1} (p - qz_{\pi(i)}z_{\pi(j)} - z_{\pi(i)}) \prod_{i=2}^{N-1} \frac{\prod_{j=i}^{N-1} z_{\pi(j)}}{1 - \prod_{j=i}^{N-1} z_{\pi(j)}}.
\end{aligned}$$

Denote by $U_N(z_1, \dots, z_N)$ the right-hand side of equation (4.36). Then

$$\begin{aligned}
& \sum_{\sigma \in \mathbb{S}_N} A_\sigma^* \prod_{i=2}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} = \\
& \sum_{k=1}^N (-1)^{N-1} (-1)^{N-k} \left(\prod_{j \neq k} (p - qz_j z_k - z_k) \frac{\prod_{j \neq k} z_j}{1 - \prod_{j \neq k} z_j} \right) U_{N-1}(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_N) \\
& = p^{\frac{(N-1)(N-2)}{2}} \sum_{k=1}^N (-1)^{k-1} \prod_{j \neq k} (p + qz_j z_k - z_k) \frac{\prod_{j \neq k} z_j}{1 - \prod_{j \neq k} z_j} \frac{\prod_{i < j, i, j \neq k} (z_j - z_i)}{\prod_{j=1, j \neq k}^N (1 - z_j)} \left(1 - \prod_{j \neq k} z_j \right)
\end{aligned}$$

where the extra $(-1)^{N-k}$ comes from the fact that all the permutations π have an extra number in the k 'th position which is permuted there from the N 'th position. The claim can then be written

$$\sum_{k=1}^N \frac{\prod_{j=1}^N (p + qz_j z_k - z_k)}{(p - qz_k) z_k \prod_{j \neq k} (z_j - z_k)} = p^{N-1} \frac{1 - \prod_{j=1}^N z_j}{\prod_{j=1}^N z_j} \quad (4.37)$$

where it was used that

$$\frac{\prod_{\substack{j > i \\ i, j \neq k}} (z_j - z_k)}{\prod_{j > i} (z_j - z_i)} = (-1)^{k-1} \frac{1}{\prod_{j \neq k} (z_j - z_k)}$$

and

$$\frac{1 - z_k}{p + qz_k^2 - z_k} = \frac{1}{p - qz_k}.$$

To show that equation (4.37) holds, consider the integral

$$\oint_{C_\infty} \frac{\prod_{j=1}^N (p + qz_j z - z)}{(p - qz) z \prod_{j=1}^N (z_j - z)}.$$

The integral is taken over a large circle so that all the poles are inside it, and since the integrand is $\mathcal{O}(z^{-2})$, the integral vanishes. The pole at each z_j gives the negation of the left side of (4.37). The pole at $z = 0$ gives $\frac{p^{N-1}}{\prod_{j=1}^N z_j}$. The last pole at $z = \frac{p}{q}$ gives $-p^{N-1}$. Thus the claim is now proven. \square

Lemma 4.9

$$\sum_{\substack{S \subset \{1, \dots, N\} \\ |S|=m}} \prod_{\substack{i \in S \\ j \in S^c}} \frac{p + qz_i z_j - z_i}{z_j - z_i} \left(1 - \prod_{j \in S^c} z_j \right) = q^m \begin{bmatrix} N-1 \\ m \end{bmatrix} \left(1 - \prod_{j=1}^N z_j \right)$$

for $N \geq m + 1$.

Proof First show two useful results. First,

$$\begin{bmatrix} N \\ m \end{bmatrix} = q^{N-m} \begin{bmatrix} N-1 \\ m-1 \end{bmatrix} + p^m \begin{bmatrix} N-1 \\ m \end{bmatrix}. \quad (4.38)$$

This is shown easily by straight calculation.

$$\begin{aligned} \begin{bmatrix} N \\ m \end{bmatrix} &= \frac{(p^N - q^N)(p^{N-1} - q^{N-1}) \cdots (p - q)}{(p^{N-m} - q^{N-m})(p^{N-m-1} - q^{N-m-1}) \cdots (p - q)(p^m - q^m) \cdots (p - q)} \\ &= \begin{bmatrix} N-1 \\ m \end{bmatrix} \frac{p^N - q^N}{p^{N-m} - q^{N-m}} = \begin{bmatrix} N-1 \\ m \end{bmatrix} \left(p^m + \frac{q^{N-m}(p^m - q^m)}{p^{N-m} - q^{N-m}} \right) \\ &= p^m \begin{bmatrix} N-1 \\ m \end{bmatrix} + q^{N-m} \begin{bmatrix} N-1 \\ m-1 \end{bmatrix} \end{aligned}$$

which proves the claim. Second, prove an easier version of the lemma. Introduce the notation $U(z_i, z_j) = \frac{p+qz_i z_j - z_i}{z_j - z_i}$. Then,

$$\sum_{\substack{S \subset \{1, \dots, N\} \\ |S|=m}} \prod_{\substack{i \in S \\ j \in S^c}} U(z_i, z_j) = \begin{bmatrix} N \\ m \end{bmatrix}. \quad (4.39)$$

This obviously holds for $N = 1$. Then assume the claim holds for $N - 1$. Now note that the left side of equation (4.39) is symmetric with respect to any z_i . It is also $\mathcal{O}(1)$ as $z_i \rightarrow \infty$ for any i . Thus, if one multiplies the left side by the Vandermonde $\prod_{j>i} (z_j - z_i)$, one obtains an antisymmetric polynomial

that is $\mathcal{O}(z_i^{N-1})$ for any i . This implies it is a polynomial of degree at most $N - 1$ in any z_i . Since it is now an antisymmetric polynomial of at most degree $N - 1$ in each z_i , it is divisible by Vandermonde, which in turn implies it must be a constant. Let this constant be $A_{N,m}$.

Set $z_N = 1$. If $N \in S$, the product of the functions U gives the factor $(\frac{p+qz_N z_j - z_N}{z_j - z_N}|_{z_N \rightarrow 1})^{|S^c|} = q^{N-m}$. If $N \notin S$, the product gives $(\frac{p+qz_i z_N - z_i}{z_N - z_i}|_{z_N \rightarrow 1})^{|S|} = p^m$. Thus

$$\begin{aligned} A_{N,m} &= q^{N-m} \sum_{\substack{S \subset \{1, \dots, N-1\} \cup \{N\} \\ |S|=m}} \prod_{\substack{i \in S \setminus \{N\} \\ j \in S^c}} U(z_i, z_j) + p^m \sum_{\substack{S \subset \{1, \dots, N-1\} \\ |S|=m}} \prod_{\substack{i \in S \\ j \in S^c \setminus \{N\}}} U(z_i, z_j) \\ &= q^{N-m} \begin{bmatrix} N-1 \\ m-1 \end{bmatrix} + p^m \begin{bmatrix} N-1 \\ m \end{bmatrix} = \begin{bmatrix} N \\ m \end{bmatrix} \end{aligned}$$

by the induction hypothesis. Then start on the actual proof of lemma 4.9. As above, notice that the left side of equation (4.9) is a polynomial of at most first degree in each z_i , $C_{N,m}(z_1, \dots, z_N)$. Similarly, the same kind of relation holds:

$$C_{N,m}(z_1, \dots, z_{N-1}, 1) = q^{N-m} C_{N-1,m-1}(z_1, \dots, z_{N-1}) + p^m C_{N-1,m}(z_1, \dots, z_{N-1}). \quad (4.40)$$

Also note that $C_{N,m}(z) = 0$ for $N = m$. Notice that the right-hand side of (4.9), call it $C'_{N,m}(z)$, satisfies the same relation (4.40). Thus also $D_{N,m}(z_1, \dots, z_N) = C_{N,m}(z_1, \dots, z_N) - C'_{N,m}(z_1, \dots, z_N)$ satisfies the relation. By the induction hypothesis, $D_{N-1,m}(z_1, \dots, z_N) = D_{N-1,m-1} = 0$. This, in turn, implies $D_{N,m}(z_1, \dots, z_{N-1}, 1) = 0$ so $D_{N,m}(z_1, \dots, z_N) = B \prod_{j=1}^N (1 - z_j)$. Solving for $C_{N,m}(z_1, \dots, z_N)$, one obtains

$$C_{N,m}(z) = C'_{N,m}(z) - B \prod_{j=1}^N (1 - z_j).$$

Thus it is sufficient to show that $B = 0$ to finish the proof. Both terms are $\mathcal{O}(z_N)$ as $z_N \rightarrow \infty$. For the sum in (4.9), if $N \in S$, the summand $\mathcal{O}(1)$. As such, consider only terms with $N \notin S$.

$$\lim_{z_N \rightarrow \infty} \frac{C_{N,m}(z)}{z_N} = -q^m \sum_{\substack{S \in \{1, \dots, N-1\} \\ |S|=m}} \prod_{\substack{i \in S \\ j \in S^c, j < N}} U(z_i, z_j) \prod_{j \in S} z_j \prod_{\substack{j \in S^c \\ j < N}} z_j = -q^m \prod_{j < N} z_j \begin{bmatrix} N-1 \\ m \end{bmatrix}$$

by induction. Similarly, this holds for $C'_{N,m}(z)$. Thus $B = 0$, or otherwise there would be an additional constant term in the asymptotics of $C_{N,m}(z)$. \square

Theorem 4.10 *The probability of finding the first particle on site x at time t is*

$$\mathbb{P}_Y(x_1(t) = x) = p^{\frac{N(N-1)}{2}} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} I(x, Y, z) d^N z \quad (4.41)$$

Proof First, note that each x_i can be written as $x_i = x + \sum_{j=1}^{i-1} k_j$. Then

$$\begin{aligned} \mathbb{P}_Y(x_1(t) = x) &= \sum_{0 < k_1, \dots, k_{N-1} \leq \infty} P_Y(x, x + k_2, x + k_2 + k_3, \dots, x_N + \sum_{i=2}^N k_i; t) \\ &= \sum_{k_2=1}^{\infty} \cdots \sum_{k_N=1}^{\infty} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} e^{\epsilon t} \prod_{\alpha=1}^N z_{\alpha}^{x-y_{\alpha}-1} \sum_{\sigma \in \mathbb{S}_N} A_{\sigma} \prod_{i=2}^N \prod_{j=i}^N z_{\sigma(j)}^{k_i} d^N z \\ &= \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} e^{\epsilon t} \prod_{\alpha=1}^N z_{\alpha}^{x-y_{\alpha}-1} \sum_{\sigma \in \mathbb{S}_N} A_{\sigma} \prod_{i=2}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{1 - \prod_{j=i}^N z_{\sigma(j)}} d^N z. \end{aligned}$$

Using lemma 4.8 on the integrand, one obtains

$$\mathbb{P}_Y(x_1(t) = x) = p^{\frac{N(N-1)}{2}} \oint \cdots \oint \prod_{i < j} \frac{z_j - z_i}{p - q z_i z_j - z_i} \frac{(1 - \prod_{j=1}^N z_j)}{\prod_{j=1}^N (1 - z_j)} e^{\epsilon t} \prod_{\alpha=1}^N z_{\alpha}^{x-y_{\alpha}-1} d^N z$$

where the integrand is exactly $I(x, Y, z)$ defined in (4.33). The proof is complete. \square

Equation (4.41) still depends explicitly on N , which is going to be a problem for the inevitable limit $N \rightarrow \infty$. One can, however, write theorem 4.10 in a form where the integrals are over contours \mathcal{C}_R that include all the poles of the integrand inside.

Theorem 4.11 *Equation (4.41) can also be written as*

$$\mathbb{P}(x_1(t) = x) = \sum_{S \subset \{1, \dots, N\}} \frac{p^{\sigma(S) - |S|}}{q^{\sigma(S) - \frac{|S|(|S|+1)}{2}}} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} I(x, Y_S, z) d^{|S|} z \quad (4.42)$$

where $\sigma(S) = \sum_{i \in S} i$, $Y_S = \{y_k : k \in S\}$, $I(x, Y_S, z)$ the integrand with only z_i , x_i for $i \in S$ and the contours are large enough circles of radius R .

Proof For the proofs of theorems 4.11 and 4.13, an incredibly convenient lemma will be stated.

Lemma 4.12 Assume $f(z_1, \dots, z_N)$ is analytic in all variables outside of origin and, for $i > k$,

$$f(z_1, \dots, z_N)|_{z_i \rightarrow \frac{z_k - p}{qz_k}} = \mathcal{O}(z_k) \text{ as } z_k \rightarrow 0$$

uniformly when all z_j , $j \neq k$, are bounded and bounded away from zero. Then, for $p, q \neq 0$,

$$\oint_{C_r} \cdots \oint_{C_r} I_f(z) d^N z = \sum_{S \subset \{1, \dots, N\}} \frac{p^{|S^c| - \sigma(S^c)}}{q^{\sigma(S) - \frac{|S|(|S|+1)}{2}}} \oint_{C_R} \cdots \oint_{C_R} I_{f,S}(z) d^{|S|} z. \quad (4.43)$$

Proof This will be proven using induction. First, note that for $N = 1$ the claim holds. Then expand the z_N contour in (4.43) to have a large enough radius to have all the poles inside it. The poles are at $z_* = 1$ and $z^* = \frac{z_k - p}{qz_k}$ for all $k < N$. Here the residue at z^* is zero when integrated over z_k as will be shown next. First note that $f(z_1, \dots, z_N)$ is $\mathcal{O}(z_k)$ as $z_k \rightarrow 0$ by assumption. Then, showing that the rest of the integrand is $\mathcal{O}(z_k^{-1})$ as $z_k \rightarrow 0$ will suffice to show that the integral vanishes.

$$\begin{aligned} & \prod_{i < j} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{1}{\prod_{j=1}^N (1 - z_j)} (p + qz_k z_N - z_k)|_{z_N \rightarrow \frac{z_k - p}{qz_k}} \\ &= \frac{\prod_{i=1}^{N-1} \left(\frac{z_k - p}{qz_k} - z_i \right)}{\prod_{\substack{i=1 \\ i \neq k}}^{N-1} (p(z_k - z_i))(1 - \frac{z_k - p}{qz_k})} \prod_{i < j}^{N-1} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{1}{\prod_{j=1}^{N-1} (1 - z_j)} \\ &= \mathcal{O}(z_k^{-1}) \end{aligned}$$

as $z_k \rightarrow 0$. Thus the residue at z^* vanishes. After expanding the N 'th integral, left side of (4.43) reads

$$\oint_{C_r} \cdots \oint_{C_r} I_f(z) d^N z = \oint_{C_r} \cdots \oint_{C_R} I_f(z) d^N z + \frac{1}{p^{N-1}} \oint_{C_r} \cdots \oint_{C_r} I_{f,S}(z) d^{|S|} z \quad (4.44)$$

where $S = \{1, \dots, N-1\}$. The factor $\frac{1}{p^{N-1}}$ comes from the fact that $\prod_{i=1}^{N-1} \frac{z_N - z_i}{p + qz_i z_N - z_i}|_{z_N \rightarrow 1} = \prod_{i=1}^{N-1} \frac{1 - z_i}{p - pz_i} = \frac{1}{p^{N-1}}$. The second term is then, by induction,

$$\frac{1}{p^{N-1}} \sum_{S \subset \{1, \dots, N-1\}} \frac{p^{|S^c| - \sigma(S^c)}}{q^{\sigma(S) - \frac{|S|(|S|+1)}{2}}} \oint_{C_R} \cdots \oint_{C_R} I_{f,S}(z) d^{|S|} z \quad (4.45)$$

where the complements are with respect to $\{1, \dots, N-1\}$. Making the substitutions $|S^c| \rightarrow |S^c| - 1$, $\sigma(S^c) \rightarrow \sigma(S^c) - N$, the p -factor from in front of the sum disappears and the complements are now with respect to $\{1, \dots, N\}$. Then consider the first term on the right side of equation (4.44). For this, define

$$\tilde{f}(z_1, \dots, z_{N-1}) = \oint_{C_R} \prod_{i < N} \frac{z_N - z_i}{p + qz_i z_N - z_i} \frac{f(z_1, \dots, z_N)}{1 - z_N} dz_N$$

so that the first term in (4.44) becomes, by induction,

$$\oint_{C_r} \cdots \oint_{C_r} I_{\tilde{f}}(z) d^{N-1} z = \sum_{\tilde{S} \subset \{1, \dots, N-1\}} \frac{p^{|\tilde{S}^c| - \sigma(\tilde{S}^c)}}{q^{\sigma(\tilde{S}) - \frac{|\tilde{S}|(|\tilde{S}|+1)}{2}}} \oint_{C_R} \cdots \oint_{C_R} I_{\tilde{f}, \tilde{S}}(z) d^{|\tilde{S}|} z \quad (4.46)$$

as long as \tilde{f} satisfies the requirements stated in the claim of the lemma. Choose $k < i < N$. Then $\frac{f(z_1, \dots, z_N)}{1 - z_N} \Big|_{z_i \rightarrow \frac{z_k - p}{qz_k}} = \mathcal{O}(z_k)$ as $z_k \rightarrow 0$ by assumption on f . Since also

$$\begin{aligned} & \prod_{j < N} \frac{z_N - z_j}{p + qz_j z_N - z_j} \Big|_{z_i \rightarrow \frac{z_k - p}{qz_k}} \\ &= \frac{z_N - z_k}{p + qz_k z_N - z_k} \frac{p + qz_k z_N - z_k}{p + z_k(qz_N - 1) + pq(z_k - z_N)} \prod_{\substack{j < N \\ j \neq i, k}} \frac{z_N - z_j}{p + qz_j z_N - z_j} = \mathcal{O}(1) \text{ as } z_k \rightarrow 0, \end{aligned}$$

\tilde{f} as a whole is $\mathcal{O}(z_k)$ as $z_k \rightarrow 0$. Thus using the induction hypothesis was deserved. It remains to show that the right-hand side of (4.43) equals the sum of (4.46) and (4.45). Write the sum in (4.43) as a sum of sums over $\tilde{S} \subset \{1, \dots, N-1\}$ and $S = \tilde{S} \cup \{N\}$. The first one is just (4.45). The second sum gives, after some manipulation, (4.46). This finishes the proof of lemma 4.12. \square .

The rest of the proof of 4.11 is a fairly straightforward application of lemma 4.12. Define

$$f(z_1, \dots, z_N) = e^{et} \left(1 - \prod_{i=1}^N z_i\right) \prod_{i=1}^N z_i^{x - y_i - 1}.$$

If f satisfies the hypothesis of the lemma, the proof is complete. The exponential stays analytic at $z_k = 0$ and the rest of the product is obviously $\mathcal{O}(z_k^{y_i - y_k})$, which is $\mathcal{O}(z_k)$ since $y_i > y_k$ due to exclusion. The proof is complete. \square .

Inspired by the first particle, one would hope that for the general particle the N -dependence vanishes for at least large contours. Indeed it turns out to be true for large contours.

Theorem 4.13 *Assuming $p \neq 0$,*

$$\mathbb{P}(x_m(t) = x) = (-1)^{m+1} (pq)^{\frac{m(m-1)}{2}} \sum_{|S| \geq m} \begin{bmatrix} |S| - 1 \\ |S| - m \end{bmatrix} \frac{p^{\sigma(S) - m|S|}}{q^{\sigma(S) - \frac{|S|(|S|+1)}{2}}} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} I(x, Y_S, z) d^{|S|} z \quad (4.47)$$

where the sum runs over all subsets $S \subset \{1, \dots, N\}$.

Proof As done earlier, let $g : \mathbb{C}^N \mapsto \mathbb{C}$ be a function such that $g(z_1, \dots, z_N)$ is analytic for all $z_i \neq 0$ with $i = 1, \dots, N$. Also assume that

$$g(z_1, \dots, z_N) \Big|_{z_i \rightarrow \frac{p}{1 - qz_k}} = \mathcal{O}(z_k^{-1}) \text{ as } z_k \rightarrow \infty,$$

uniformly for all $z_j, j \neq k$ bounded and bounded away from the origin. Recall from (4.34) and (4.35) that

$$I_g(z) = \prod_{j > i} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{g(z_1, \dots, z_N)}{\prod_{i=1}^N (1 - z_i)} \quad (4.48)$$

and

$$I_{g,S}(z) = \prod_{\substack{j > i \\ i, j \in S}} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{g(z_1, \dots, z_N) \Big|_{z_i \rightarrow 1, i \in S^c}}{\prod_{i \in S} (1 - z_i)} \quad (4.49)$$

for any subset $S \subset \{1, \dots, N\}$. Identically to the proof of 4.10, some important results from which the result almost directly follows will be proven first. The following result is analogous to lemma 4.12.

Lemma 4.14 For the function g defined as above and $p, q \neq 0$,

$$\oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} I_g(z) d^N z = \sum_{S \subset \{1, \dots, N\}} (-1)^{|S^c|} \frac{q^{\sigma(S^c) - N|S^c|}}{p^{\frac{|S|(|S^c| + N + 1)}{2} - \sigma(S)}} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} I_{g,S}(z) d^{|S|} z. \quad (4.50)$$

Proof In lemma 4.12, exchange p and q and use it with $f(z_1, \dots, z_N) = g(z_N^{-1}, \dots, z_1^{-1}) \prod_{j=1}^N z_j^{-1}$.

Obviously (almost by definition), f satisfies the hypothesis in the lemma due to the way g is defined. Then one can see by using lemma 4.12 that the left side of (4.43) with p and q interchanged equals

$$\begin{aligned} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} I_f(z) d^N z &= \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} \prod_{j>i} \frac{z_j - z_i}{q + p z_i z_j - z_i} \frac{g(z_N^{-1}, \dots, z_1^{-1}) \prod_{j=1}^N z_j^{-1}}{\prod_{j=1}^N (1 - z_j)} d^N z \\ &= \sum_{S \subset \{1, \dots, N\}} \frac{q^{|S^c| - \sigma(S^c)}}{p^{\sigma(S) - \frac{|S|(|S|+1)}{2}}} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \prod_{\substack{i,j \in S \\ j>i}} \frac{z_j - z_i}{q + p z_i z_j - z_i} \frac{g(z_N^{-1}, \dots, z_1^{-1}) \prod_{j=1}^N z_j}{\prod_{i \in S} (1 - z_i)} d^{|S|} z. \end{aligned}$$

This, after a change of variables $z_j \rightarrow \frac{1}{z_{N-j+1}}$, becomes

$$\begin{aligned} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} I_f(z) d^N z &= \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \prod_{j>i} \frac{z_j - z_i}{p + q z_i z_j - z_i} \frac{g(z_1, \dots, z_N)}{\prod_{j=1}^N (z_j - 1)} d^N z \\ &= (-1)^N \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} I_g(z) d^N z = \sum_{S \subset \{1, \dots, N\}} (-1)^{|S|} \frac{q^{|S^c| - \sigma(S^c)}}{p^{\sigma(S) - \frac{|S|(|S|+1)}{2}}} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} I_{g,\tilde{S}}(z) d^N z \end{aligned}$$

where $\tilde{S} = \{N - i + 1 : i \in S\}$. The very right side of this equation equals the right side of (4.50). This finishes the proof. \square

For the next lemma, introduce some more notation. For $T \subset U$ where $T, U \subset \{1, \dots, N\}$, write $\sigma(T, U)$ = sum of positions of elements of T in U . Define the sign of a set U as

$$\text{sgn}(U) = (-1)^{\#\{(i,j): i>j, i \in U, j \in U^c\}}.$$

Then assume $T, U \subset \{1, \dots, N\}$ are disjoint subsets. Let

$$I(x, Y_{T,U}, z) = (1 - \prod_{j \in U} z_j) \frac{\prod_{\substack{i,j \in U \text{ or } i,j \in T \\ j>i}} (z_j - z_i)}{\prod_{j \in T \cup U} (1 - z_j)} \frac{\prod_{i \in T, j \in U} (p + q z_i z_j - z_i)}{\prod_{\substack{i,j \in T \cup U \\ j>i}} (p + q z_i z_j - z_i)} \prod_{j \in T \cup U} z_j^{x - y_j - 1} e^{\epsilon t}.$$

Lemma 4.15 For $p, q \neq 0$, it holds that

$$\begin{aligned} \mathbb{P}(x_m(t) = x) &= p^{\frac{(N-m)(N-m+1)}{2} + \frac{m(m-1)}{2}} q^{\frac{(m-1)(m-2)}{2}} \sum_{|U|=m-1, U \subset \{1, \dots, N\}} \text{sgn}(U) \\ &\times \sum_{T \subset U} (-1)^{|T| + \sigma(U \setminus T) - \sigma(U \setminus T, U)} \frac{q^{\sigma(U \setminus T) - (m-1)|U \setminus T|}}{p^{\frac{\sigma(U \setminus T) + |T|(m+|U \setminus T|)}{2}}} \oint_{\mathcal{C}_r} \cdots \oint_{\mathcal{C}_r} I(x, Y_{T,U^c}, z) d^{|T \cup U^c|} z \end{aligned} \quad (4.51)$$

Proof As in the beginning of the proof of theorem 4.10, write

$$X = \{x - v_{m-1} - \dots - v_1, x - v_{m-1} - \dots - v_1, \dots, x - v_1, x, x + w_1, \dots, x + w_1 + \dots + w_{N-m}\}.$$

The left side of equation (4.51) can be obtained by summing left side of equation (4.27) over all w_i and v_i . Since the contours in (4.27) are small, $z_j^{w_i}$ are positive and inside the unit circle for all $i, j = 1, \dots, N$. Thus the w_i sums are summable and the probability becomes

$$\begin{aligned} \mathbb{P}(x_m(t) = x) &= \sum_{1 \leq v_1, \dots, v_{m-1} \leq \infty} \oint_{C_r} \dots \oint_{C_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma e^{\epsilon t} \left(\prod_{j=1}^N z_j^{x-y_j-1} \right) z_{\sigma(1)}^{-v_1-\dots-v_{m-1}} \dots z_{\sigma(m-1)}^{-v_1} \\ &\times \frac{z_{\sigma(m+1)} z_{\sigma(m+2)}^2 \dots z_{\sigma(N)}^{N-m}}{(1 - z_{\sigma(m+1)}) \dots (1 - z_{\sigma(N)})}. \end{aligned}$$

To sum over v_i , move the $z_{\sigma(i)}$ contours out for $i = 1, \dots, m-1$ in that order. For any i , there are poles at

$$z_{\sigma(i)} = \begin{cases} \frac{z_{\sigma(k)} - p}{q z_{\sigma(k)}} & \text{if } k > i \\ \frac{p}{1 - q z_{\sigma(k)}} & \text{if } k < i. \end{cases}$$

For $z_{\sigma(1)}$ the poles are at points $\frac{z_{\sigma(k)} - p}{q z_{\sigma(k)}}$ for all $k > 1$. These are all far from the origin since the contour for each $z_{\sigma(k)}$ is small. Thus, one can take the $z_{\sigma(1)}$ contour to be large, yet still not past the poles. Similarly, for $z_{\sigma(j)}$, all the poles with $k > j$ are large. Note that for $z_{\sigma(2)}$, there is an additional pole at $z_{\sigma(2)}^* = \frac{p}{1 - q z_{\sigma(1)}}$. Now claim that the residue at $z_{\sigma(2)}^*$ gives zero when integrated over $z_{\sigma(1)}$. First, note that $z_{\sigma(2)}^* = \mathcal{O}(z_{\sigma(1)}^{-1})$ as $z_{\sigma(1)} \rightarrow \infty$, that the $z_{\sigma(1)}$ contour is large and $z_{\sigma(2)}$ is analytic in $z_{\sigma(1)}$ outside the $z_{\sigma(1)}$ contour. All the factors in the integrand combined make the integral vanish as can be seen as follows.

First, $A_\sigma = \mathcal{O}(1)$ at infinity. The product

$$z_{\sigma(1)}^{-v_1-\dots-v_{m-1}-x-1-y_{\sigma(1)}} \left(\frac{p}{1 - q z_{\sigma(2)}} \right)^{-v_1-\dots-v_{m-2}-x-1} z_{\sigma(1)} \left(\frac{p}{1 - q z_{\sigma(1)}} \right)^{-y_{\sigma(2)}}$$

is analytic outside of the $z_{\sigma(1)}$ contour and as $z_{\sigma(1)}$ goes to infinity, the product is $\mathcal{O}(z_{\sigma(1)}^{-v_{m-1}-y_{\sigma(1)}+y_{\sigma(2)}})$. The exponent of $z_{\sigma(1)}$ in this is ≤ 2 since $v_{m-1} \geq 1$ and $y_{\sigma(1)} > y_{\sigma(2)}$. Finally, $e^{\epsilon t}$ adds nothing to the integral since the part of the exponent with $z_{\sigma(1)}$ and $z_{\sigma(2)}$ is

$$\frac{p}{z_{\sigma(1)}} + q z_{\sigma(1)} + (1 - q z_{\sigma(1)}) + \frac{pq}{1 - q z_{\sigma(1)}} \rightarrow 1$$

as $z_{\sigma(1)} \rightarrow \infty$. Thus the integral is zero.

Repeating the above process for all the desired contours allows one to sum over all the v_i without convergence issues. The probability now reads

$$\begin{aligned} \mathbb{P}(x_m(t) = x) &= \oint_{C_r} \dots \oint_{C_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma e^{\epsilon t} \left(\prod_{j=1}^N z_j^{x-y_j-1} \right) \frac{1}{(z_{\sigma(1)} - 1) \dots (z_{\sigma(1)} z_{\sigma(2)} \dots z_{\sigma(m-1)} - 1)} \\ &\times \frac{z_{\sigma(m+1)} z_{\sigma(m+2)}^2 \dots z_{\sigma(N)}^{N-m}}{(1 - z_{\sigma(m+1)}) \dots (1 - z_{\sigma(N)})}. \end{aligned}$$

Write

$$\begin{aligned}
A_\sigma &= sgn(\sigma) \frac{\prod_{j>i} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)})}{\prod_{j>i} (p + qz_i z_j - z_i)} \\
&= sgn(\sigma) \frac{\prod_{i<j<m} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}) \prod_{m \leq i < j} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}) \prod_{i < m \leq j} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)})}{\prod_{j>i} (p + qz_i z_j - z_i)}.
\end{aligned}$$

Thus the integral can be organized as

$$\begin{aligned}
\mathbb{P}(x_m(t) = x) &= \oint \cdots \oint \frac{e^{\epsilon t}}{\prod_{j>i} (p + qz_i z_j - z_i)} \sum_{\sigma \in \mathbb{S}_N} sgn(\sigma) \\
&\quad \times \frac{\prod_{i<j<m} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)})}{(z_{\sigma(1)} - 1)(z_{\sigma(1)}z_{\sigma(2)} - 1) \cdots (z_{\sigma(1)} \cdots z_{\sigma(m-1)} - 1)} \\
&\quad \times \frac{(\prod_{m \leq i < j} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}))z_{\sigma(m+1)}z_{\sigma(2)}^2 \cdots z_{\sigma(N)}^{N-m}}{(1 - z_{\sigma(m+1)}z_{\sigma(m+2)} \cdots z_{\sigma(N)}) \cdots (1 - z_{\sigma(N)})} \\
&\quad \times \prod_{i < m \leq j} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}) d^N z.
\end{aligned}$$

where the integrals are over \mathcal{C}_r for z_i where $i \in U^c$ and over \mathcal{C}_R for z_i where $i \in U$. Choose some subset $U \subset \{1, \dots, N\}$ for which $|U| = m - 1$. Take the part of the sum for which $\sigma(i) \in U$ with $i \leq m - 1$. Then the last product in the integrand equals

$$\prod_{i < m \leq j} (p + qz_{\sigma(1)}z_{\sigma(j)} - z_{\sigma(i)}) = \prod_{\substack{i \in U \\ j \in U^c}} (p + qz_i z_j - z_i)$$

which is independent of σ . Also note that $sgn(\sigma) = sgn(U)sgn(\sigma_1)sgn(\sigma_2)$ where σ_1 and σ_2 are defined by writing the sum over $\sigma \in \mathbb{S}_N$ as the sum over $|U| = m - 1$, $U \subset \{1, \dots, N\}$ of two sums, the first over $\sigma_1 : \{1, \dots, m - 1\} \mapsto U$ and the second over $\sigma_2 : \{m, \dots, N\} \mapsto U^c$. That is, change $\sum_{\sigma \in \mathbb{S}_N} \rightarrow \sum_{\substack{U \subset \{1, \dots, N\} \\ |U|=m-1}} \sum_{\sigma_1} \sum_{\sigma_2}$. Both these sums are of the form of identity 4.8, either as is or after

interchanging p and q and changing variables by $z_i \rightarrow \frac{1}{z_{N-i+1}}$. That is,

$$\begin{aligned}
& \sum_{\sigma_1} \text{sgn}(\sigma_1) \frac{\prod_{i < j < m} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)})}{\prod_{i=1}^{m-1} \prod_{j=1}^i (z_{\sigma(j)} - 1)} = q^{\frac{(m-1)(m-1)}{2}} \frac{\prod_{\substack{j > i \\ i, j \in U}} (z_j - z_i)}{\prod_{j \in U} (z_j - 1)}, \\
& \sum_{\sigma_2} \text{sgn}(\sigma_2) \prod_{m \leq i < j} (p + qz_{\sigma(i)}z_{\sigma(j)} - z_{\sigma(i)}) \prod_{i=m+1}^N \frac{\prod_{j=i}^N z_{\sigma(j)}}{(1 - \prod_{j=i}^N z_{\sigma(j)})} \\
& = p^{\frac{(N-m)(N-m-1)}{2}} (1 - \prod_{j \in U^c} z_j) \frac{\prod_{\substack{j > i \\ i, j \in U^c}} (z_j - z_i)}{\prod_{j \in U^c} (1 - z_j)}.
\end{aligned}$$

What is left is to write all the integrals over small contours again. This is achieved by using lemma 4.14. Use the lemma with

$$\begin{aligned}
N & \rightarrow m-1 \\
\{1, \dots, N\} & \rightarrow U \\
S & \rightarrow T \subset U.
\end{aligned}$$

That is, use

$$\oint_{\mathcal{C}_R} \dots \oint_{\mathcal{C}_R} I_g(z) d^{m-1}z = \sum_{T \subset U} (-1)^{|U \setminus T|} \frac{q^{\sigma(U \setminus T, U) - (m-1)|U \setminus T|}}{p^{\frac{|T|(m+|U \setminus T|)}{2} - \sigma(T, U)}} \oint_{\mathcal{C}_r} \dots \oint_{\mathcal{C}_r} I_{g,T}(z) d^{|T|}z \quad (4.52)$$

on the probability

$$\begin{aligned}
\mathbb{P}(x_m(t) = x) & = q^{\frac{(m-1)(m-2)}{2}} p^{\frac{(N-m)(N-m-1)}{2}} \sum_{\substack{U \subset \{1, \dots, N\} \\ |U| = m-1}} \text{sgn}(U) (-1)^{|U|} \\
& \times \oint \dots \oint \frac{(1 - \prod_{j \in U^c} z_j) \prod_{\substack{i, j \in U^c \\ j > i}} (z_j - z_i) \prod_{j \in U^c} (z_j^{x-y_j-1} e^{\epsilon(z_j)t})}{\prod_{j \in U^c} (1 - z_j) \prod_{\substack{i, j \in U^c \\ j > i}} (p + qz_i z_j - z_i)} \\
& \times \frac{\prod_{j > i, i, j \in U} (z_j - z_i) \prod_{j \in U} z_j^{x-y_j-1} e^{\epsilon(z_j)t}}{\prod_{j \in U} (1 - z_j) \prod_{\substack{i, j \in U \\ j > i}} (p + qz_i z_j - z_i)} \prod_{\substack{i \in U, j \in U^c \\ j > i}} (p + qz_i z_j - z_i) d^N z
\end{aligned} \quad (4.53)$$

with

$$g(\{z_i\}_{i \in U}) = \prod_{j \in U} (z_j^{x-y_j-1} e^{\epsilon(z_j)t}) \oint_{\mathcal{C}_r} \dots \oint_{\mathcal{C}_r} \prod_{\substack{i > j \\ i \in U, j \in U^c}} \frac{p + qz_i z_j - z_i}{p + qz_i z_j - z_j} \prod_{j \in U^c} dz_j.$$

To use equation (4.52), g needs to satisfy the hypothesis in lemma 4.14.

The poles of g outside of the origin are at points $z_i = \frac{z_j - p}{qz_j}$ for $j \in U^c$. Since z_j are small, the poles are well outside \mathcal{C}_R , and thus g is analytic for all $z_i \neq 0$. Now set $z_i \rightarrow \frac{p}{1 - qz_k}$ for some $i < k$. Then the product $z_i^{x-y_i-1} z_k^{x-y_k-1} = \mathcal{O}(z_k^{y_i-y_k}) = \mathcal{O}(z_k^{-1})$ as $z_k \rightarrow \infty$, as before. Similarly, the exponents involving i and k are $1 - qz_k + \frac{pq}{1 - qz_k} + \frac{p}{z_k} + qz_k$ which is bounded at infinity. Fix $j \in U^c$, $j > k$. Then the pole $z_j = \frac{p}{1 - qz_k}$ passes the z_j contour on the way to infinity. The residue, however, is $\mathcal{O}(1)$, as are the other factors in the integrand. Thus, repeating for each j results in a sum of integrals that are all $\mathcal{O}(1)$. The integral stays bounded, so g satisfies the hypothesis.

Consider the right side of equation (4.52). The factor $g(\{z_i\}_{i \in U})|_{z_j \rightarrow 1, j \in U \setminus T}$ yields some additional factors to the summand before the integral. First, note that $\frac{p + qz_j - 1}{p + qz_j - z_j} = -\frac{q}{p}$. Obviously there are $|U \setminus T|$ z_j being set to unity. Each such j in the product has j minus the position of j in U that satisfy $j > i$ and $i \in U^c$. The exponent of $-\frac{q}{p}$ is the sum of this number summed over all $i \in U \setminus T$ and equals $\sigma(U \setminus T) - \sigma(U \setminus T, U)$. The rest of $I_{g,T}(z)$ is just $I(x, Y_{T,U}, z)$. Thus a straightforward application of equation (4.52) to equation (4.53) finishes the proof of 4.15. \square

Returning to the proof of theorem 4.13, assume for now that $q \neq 0$. Take some fixed $S \subset \{1, \dots, N\}$ with $|S^c| \leq m$. In lemma 4.15, sum over all $T, U \subset \{1, \dots, N\}$ such that $T \subset U$ and $T \cup U^c = S$. Then look into finding everything in terms of T and U^c . First, since the position of i in U is $\#\{j | j \leq i, j \in U\}$, it follows that

$$\sigma(T, U) = \#\{(i, j) | i \geq j, i \in T, j \in U\}.$$

Looking at the two terms with σ in the power of (-1) in the summand,

$$\begin{aligned} \sigma(U \setminus T, U) &= \#\{(i, j) | i \geq j, i \in U \setminus T, j \in U\}, \\ \sigma(U \setminus T) &= \#\{(i, j) | i \geq j, i \in U \setminus T, j \in U^c\}. \end{aligned}$$

Thus

$$\sigma(U \setminus T) - \sigma(U \setminus T, U) = \#\{(i, j) | i \geq j, i \in U \setminus T, j \in U^c\}.$$

Recall that $\text{sgn}(U) = (-1)^{\#\{(i, j) | i > j, i \in U, j \in U^c\}}$. Then the combined power of (-1) is

$$|T| + \#\{(i, j) | i > j, i \in T, j \in U^c\} + \#\{(i, j) | i \geq j, i \in U \setminus T, j \in U^c\} = |T| + \#\{(i, j) | i > j, i \in T, j \in U^c\}.$$

Note that $U \setminus T = (U^c \cup T)^c = S^c$. Thus the power of $\frac{q}{p}$ is

$$\frac{q^{\sigma(S^c) - (m-1)|S^c|}}{p^{\sigma(S^c) + \frac{|T|(m+|S^c|)}{2}}}.$$

In the integrand, write

$$\prod_{\substack{i, j \in U^c \text{ or } i, j \in T \\ j > i}} (z_j - z_i) = \frac{\prod_{\substack{i, j \in U^c \cup T \\ j > i}} (z_j - z_i)}{\prod_{\substack{i \in U^c, j \in T \\ j > i}} (z_j - z_i) \prod_{\substack{i \in T, j \in U^c \\ j > i}} (z_j - z_i)} = (-1)^{\#\{(i, j) | i > j, i \in T, j \in U^c\}} \frac{\prod_{\substack{i, j \in U^c \cup T \\ j > i}} (z_j - z_i)}{\prod_{\substack{i \in T \\ j \in U^c \\ j > i}} (z_j - z_i)}.$$

This cancels with the earlier power of (-1) and leaves just $(-1)^{|T|}$. The integrand is then

$$(-1)^{|T|} \left(\prod_{\substack{i \in T \\ j \in U^c}} \frac{p + qz_j z_i - z_i}{z_j - z_i} \left(1 - \prod_{j \in U^c} z_j \right) \right) \frac{\prod_{\substack{i, j \in T \cup U^c \\ j > i}} (z_j - z_i)}{\prod_{i \in T \cup U^c} (1 - z_i) \prod_{\substack{i, j \in T \cup U^c \\ j > i}} (p + qz_j z_i - z_i)} \prod_{j \in T \cup U^c} (z_j^{x-y_j-1} e^{\epsilon(z_j)t})$$

Taking a fixed $S \subset \{1, \dots, N\}$ with $|S^c| < m$ and in lemma 4.15 sum over all $T, U \subset \{1, \dots, N\}$ such that $T \cup U^c = S$. Then $|U| = m - 1$ becomes $|T| = m - 1 - |S^c|$. The first part of the integrand along with the new sum is exactly identity (4.9) with

$$\begin{aligned} \{1, \dots, N\} &\rightarrow S \\ S &\rightarrow T \\ m &\rightarrow m - 1. \end{aligned}$$

Overall, the equation is now

$$\begin{aligned} \mathbb{P}(x_m(t) = x) = & p^{\frac{(N-m)(N-m+1)}{2}} q^{\frac{m(m-1)}{2}} \sum_{\substack{S \subset \{1, \dots, N\} \\ |S^c| < m}} (-1)^{m-1-|S^c|} \left[\frac{|S| - 1}{m - 1 - |S^c|} \right] \frac{q^{\sigma(S^c) - m|S^c|}}{p^{\sigma(S^c) - \frac{|S^c|(|S^c|+1)}{2}}} \\ & \times \oint_{\mathcal{C}_r} \dots \oint_{\mathcal{C}_r} I(x, Y_S, z) d^{|S|} z. \end{aligned} \quad (4.54)$$

This is fairly close to the hypothesis, but still has small contours and dependence on N .

Denote by $\tilde{\mathbb{P}}(x_{N-m+1}(t) = -x)$ the particle's position with initial condition $\tilde{Y} = \{-y_N, \dots, y_1\}$. Then note that $\mathbb{P}(x_m(t) = x) = \tilde{\mathbb{P}}(x_{N-m+1}(t) = -x)$ with their respective initial conditions and p and q interchanged. Thus in equation (4.54) replace $m \rightarrow N - m + 1$ and $S \rightarrow \tilde{S} = \{N - i + 1 | i \in S\}$. Also,

$$\begin{aligned} |\tilde{S}^c| &= N - |\tilde{S}| \\ \sigma(\tilde{S}) &= \sum_{i \in \tilde{S}} (N - i + 1) = (N + 1)|S| - \sigma(S). \end{aligned}$$

The contours are enlarged by a change of variables $z_i \rightarrow \frac{1}{z_{N-i+1}}$. This proves theorem 4.13. \square

Theorem 4.13 has a simple corollary more suitable for later analysis.

Corollary 4.16 *For step initial condition, i.e. $Y = \mathbb{Z}^+$, equation (4.47) takes the form*

$$\begin{aligned} \mathbb{P}(x_m(t) = x) = & (-1)^{m+1} q^{\frac{m(m-1)}{2}} \sum_{k \geq m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} p^{(k-m)(k-m+1)/2} q^{\frac{k(k+1)}{2}} \\ & \oint_{\mathcal{C}_R} \dots \oint_{\mathcal{C}_R} \prod_{j \neq i} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{1 - \prod_{j=1}^k z_j}{\prod_{j=1}^k (1 - z_j)(qz_j - p)} \prod_{j=1}^k z_j^{x-1} e^{\epsilon t} d^k z \end{aligned} \quad (4.55)$$

Proof In equation (4.47), sum over all $S \subset \mathbb{Z}^+$ with $|S| = k$, i.e. $\sum_{|S| \geq m} \rightarrow \sum_{k \geq m} \sum_{\substack{S \subset \mathbb{Z}^+ \\ |S|=k}}$ and rearrange the

variables by

$$z_{s_1} \rightarrow z_1, z_{s_2} \rightarrow z_2, \dots, z_{s_k} \rightarrow z_k$$

where $S = \{s_1, \dots, s_k\}$. Since $\sigma(S) = \sum_{j=1}^k s_j$, the probability can be written

$$\begin{aligned} \mathbb{P}(x_m(t) = x) &= (-1)^{m+1} (pq)^{\frac{m(m-1)}{2}} \sum_{k \geq m} \sum_{\substack{S \subset \mathbb{Z}^+ \\ |S|=k}} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} \frac{p^{-mk}}{q^{-\frac{k(k+1)}{2}}} \\ &\times \oint_{C_R} \dots \oint_{C_R} \frac{\prod_{j>i} (z_j - z_i)}{\prod_{i \neq j} (p + qz_i z_j - z_i)} \frac{1 - z_1 \dots z_k}{\prod_{j=1}^k (1 - z_j)} \prod_{j=1}^k (z_j^{x-1} e^{\epsilon(z_j)t}) \\ &\times \prod_{j<i} (p + qz_i z_j - z_i) \prod_{j=1}^k \left(\frac{q}{p} z_j\right)^{-s_j} d^k z \end{aligned}$$

Since the exponents of z_i are negative and the contours large, do the sum over $0 < z_1 < \dots < z_k$. This results in the last factor in the integrand being written as

$$\frac{\prod_{j>i} (p + qz_i z_j - z_i)}{\left(\left(\frac{q}{p} z_1\right)\left(\frac{q}{p} z_2\right) \dots \left(\frac{q}{p} z_k\right) - 1\right) \left(\frac{q}{p} z_2 \dots \frac{q}{p} z_k - 1\right) \dots \left(\frac{q}{p} z_k - 1\right)}.$$

The rest of the integrand aside from the Vandermonde determinant is symmetric, so the integral remains unchanged in an antisymmetrization of the above factor. With a change of variables $z_j = \frac{p}{q} \eta_{k-j+1}$ the factor becomes

$$\frac{1}{k!} \left(\frac{p}{q}\right)^{\frac{k(k-1)}{2}} \sum_{\sigma \in \mathbb{S}_k} \text{sgn}(\sigma) \frac{\prod_{j>i} (q + p\eta_{\sigma(i)} \eta_{\sigma(j)} - \eta_{\sigma(i)})}{(\eta_{\sigma(1)} - 1)(\eta_{\sigma(1)} \eta_{\sigma(2)} - 1) \dots (\eta_{\sigma(1)} \eta_{\sigma(2)} \dots \eta_{\sigma(k)} - 1)}.$$

This is exactly of the form in identity 4.8 with p and q interchanged and the change of variables $z_i \rightarrow \frac{1}{z_{N-i+1}}$. It then becomes

$$\frac{1}{k!} \frac{p^{k(k-1)}}{q^{\frac{k(k-1)}{2}}} \frac{\prod_{i<j} (\eta_j - \eta_i)}{\prod_{j=1}^k (\eta_i - 1)} = \frac{p^{\frac{k(k-1)}{2}}}{k!} \frac{\prod_{i>j} (z_j - z_i)}{\prod_{j=1}^k (qz_j - p)}.$$

which proves the claim. \square

4.3.3 Determinantal form

It turns out that corollary 4.16 leaves the one-point function in a form optimal for further analysis. It should be noted that we have already assumed a specific initial condition, $Y = \mathbb{Z}^+$.

Theorem 4.17 For $p, q \neq 0$,

$$\mathbb{P}(x_m(t) \leq x) = \oint_{C_R} \frac{\det(I - \lambda q K)}{\prod_{j=0}^{m-1} (1 - \lambda \tau^j)} \frac{d\lambda}{\lambda} \quad (4.56)$$

where $\tau = \frac{p}{q}$ and K is an operator acting on $L^2(\mathcal{C}_R)$:

$$Kf(z) = \oint_{\mathcal{C}_R} K(z, z')f(z')dz'$$

with kernel

$$K(z, z') = \frac{z^x e^{\epsilon t}}{p + qzz' - z}. \quad (4.57)$$

Proof First, sum (4.55) over x going from $-\infty$ to x . The contours are large (at least $R > 1$), so the sum is finite and equals

$$\begin{aligned} \mathbb{P}(x_m(t) \leq x) = & (-1)^m q^{\frac{m(m-1)}{2}} \sum_{k \geq m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} p^{(k-m)(k-m+1)/2} q^{\frac{k(k+1)}{2}} \\ & \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \prod_{j \neq i} \frac{z_j - z_i}{p + qz_i z_j - z_i} \frac{\prod_{j=1}^k z_j^x}{\prod_{j=1}^k (1 - z_i)(qz_i - p)} e^{\epsilon t} d^k z \end{aligned} \quad (4.58)$$

since $\sum_{x'=-\infty}^x \prod_{j=1}^k z_j^{x'-1} = -\frac{\prod_{j=1}^k z_j^x}{1 - \prod_{j=1}^k z_j}$. The following lemma can be used to write the integrand as a determinant.

Lemma 4.18

$$\det\left(\frac{1}{p + qz_i z_j - z_i}\right)_{1 \leq i, j \leq k} = (-1)^k (pq)^{\frac{k(k-1)}{2}} \prod_{i \neq j} \frac{z_j - z_i}{p + qz_i z_j - z_i} \prod_{j=1}^k \frac{1}{(1 - z_j)(qz_j - p)}$$

Proof First, make a change of variables on the function inside the determinant with $z_i = \frac{\eta_i + 1}{\eta_i + \tau - 1}$. This has the effect that

$$\frac{1}{p + qz_i z_j - z_i} = -\frac{1}{p(1 - \tau)} \frac{(1 + \tau\eta_i)(1 + \tau\eta_j)}{\eta_i - \tau\eta_j}.$$

As is known from the theory of determinants, a Cauchy determinant is of the form

$$\det\left(\frac{1}{\eta_i - \tau\eta_j}\right) = \frac{\prod_{j>i} (\eta_i - \eta_j)(\tau\eta_j - \tau\eta_i)}{\prod_{i,j=1}^k (\eta_i - \tau\eta_j)} = \frac{\tau^{\frac{k(k-1)}{2}} (-1)^{k-1} \prod_{i \neq j} (\eta_i - \eta_j)}{(1 - \tau)^k \prod_{j=1}^k \eta_j \prod_{i \neq j} (\eta_i - \tau\eta_j)}.$$

From this it follows that

$$\det\left(\frac{1}{p + qz_i z_j - z_i}\right) = (-1)^k \frac{\tau^{\frac{k(k-1)}{2}}}{p^k (1 - \tau)^{2k}} \prod_{j=1}^k \frac{(1 - \tau\eta_j)^2}{\eta_j} \prod_{i \neq j} \frac{\eta_i - \eta_j}{\eta_i - \tau\eta_j}.$$

Doing the change of variables back to z with $\eta_i = \frac{1}{p} \frac{qz_i - p}{1 - z_i}$ proves the claim with

$$\begin{aligned} \frac{(1 + \tau\eta_i)^2}{\eta_i} &= \frac{p(1 - \tau)^2}{(1 - z_i)(qz_i - p)} \\ \frac{\eta_i - \eta_j}{\eta_i - \tau\eta_j} &= q \frac{z_j - z_i}{p + qz_i z_j - z_i}. \end{aligned}$$

□

Lemma 4.18 can be used to write equation (4.58) with a determinant as the integrand.

$$\mathbb{P}(x_m(t) \leq x) = (-1)^m (pq)^{\frac{m(m-1)}{2}} \sum_{k \geq m} \frac{(-1)^k}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} \frac{(pq)^k}{p^{km}} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \det(K(z_i, z_j))_{1 \leq i, j \leq k} d^k z \quad (4.59)$$

where K is the kernel (4.57). Then note that for large λ [19, p. 26],

$$(-1)^m (pq)^{\frac{m(m-1)}{2}} \sum_{k \geq m} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} \frac{(pq)^k}{p^{km}} \lambda^{-k} = \prod_{j=1}^m \frac{1}{1 - \frac{\lambda p^{j-1}}{q^j}} \quad (4.60)$$

and the expansion of the Fredholm determinant is

$$\det(I - \lambda K) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \det(K(z_i, z_j))_{1 \leq i, j \leq k} d^k z. \quad (4.61)$$

Multiplying (4.60) and (4.61) together with λ^{-1} and integrating over a large contour yields

$$\begin{aligned} & \oint_{\mathcal{C}_R} \frac{\det(I - \lambda q K)}{\prod_{j=0}^{m-1} (1 - \lambda \tau^j)} \frac{d\lambda}{\lambda} \\ &= \oint_{\mathcal{C}_R} (-1)^m (pq)^{\frac{m(m-1)}{2}} \sum_{k \geq m} \sum_{k'=0}^{\infty} \frac{(-1)^{k'}}{(k')!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} \frac{(pq)^k}{p^{km}} \lambda^{k'-k-1} \oint_{\mathcal{C}_R} \cdots \oint_{\mathcal{C}_R} \det(K(z_i, z_j))_{1 \leq i, j \leq k'} d^{k'} z d\lambda. \end{aligned}$$

The integral on the right-hand side gives the condition $k = k'$, for which the right-hand side becomes (4.59) which proves the claim. □

4.3.4 Asymptotic analysis

With the help of equation (4.56) one can now proceed to do asymptotic analysis on ASEP. The following theorem is Theorem 3 of [20] and will be proved in more detail here. When defining a kernel $L(\eta, \eta')$, it is always assumed it is the kernel of some operator L .

Theorem 4.19 *When $0 \leq p < q$,*

$$\lim_{t \rightarrow \infty} \mathbb{P}\left(\frac{x_m(\frac{t}{\gamma}) - c_1 t}{c_2 t^{\frac{1}{3}}} \leq s\right) = F_2(s) \quad (4.62)$$

uniformly for σ in a compact subset of $(0, 1)$. Here $\sigma = \frac{m}{t}$, $c_1 = -1 + 2\sqrt{\sigma}$, $c_2 = \sigma^{-\frac{1}{6}}(1 - \sqrt{\sigma})^{\frac{2}{3}}$, $\gamma = q - p$ and F_2 is the Tracy-Widom distribution.

Proof Define

$$\begin{aligned}\phi(\eta) &= \left(\frac{1-\tau\eta}{1-\eta}\right)^x e^{(\frac{1}{1-\eta}-\frac{1}{1-\tau\eta})t}, \\ \phi_n(\eta) &= \prod_{k=0}^{n-1} \phi(\tau^k \eta), \\ \phi_\infty(\eta) &= (1-\eta)^{-x} e^{\frac{\eta}{1-\eta}t}, \\ K_1(\eta, \eta') &= \frac{\phi(\tau\eta)}{\eta' - \tau\eta}, \\ K_2(\eta, \eta') &= \frac{\phi(\eta')}{\eta' - \tau\eta},\end{aligned}$$

where both kernels K operate on γ , a small clockwise circle around $\eta = 1$. Here $K_2(\eta, \eta')$ is kernel (4.57) after substitutions $\xi = \frac{1-\tau\eta}{1-\eta}$, $\xi' = \frac{1-\tau\eta'}{1-\eta'}$. The proof uses a series of lemmas which will be proven next.

Lemma 4.20 *When $s \mapsto \Gamma_s$ is a deformation of closed curves and $L(\eta, \eta')$ is a kernel analytic in a neighborhood of $\Gamma_s \times \Gamma_s \subset \mathbb{C}^2$ for all s , the Fredholm determinant of L acting on Γ_s does not depend on s .*

Proof First note that from the definition of the Fredholm determinant, $\det(I - \lambda L)$ is a series that is determined up to constants by $Tr(L^n)$. Thus one only needs to consider the traces. Note that [21]

$$Tr(L^n) = \oint_{\Gamma_s} \cdots \oint_{\Gamma_s} L(\eta_1, \eta_2) \cdots L(\eta_{n-1}, \eta_n) L(\eta_n, \eta_1) d\eta_1 \cdots d\eta_n. \quad (4.63)$$

Now assume s' is in some small enough neighborhood of s . Then the contours in equation (4.63) may be replaced with $\Gamma_{s'}$, obtaining the trace of L acting on $\Gamma_{s'}$ which equals the trace of L acting on Γ_s . Thus the map $s \mapsto Tr_{\Gamma_s}(L^n)$ is locally constant in s . From basic complex analysis it follows that the map is constant in s .

Lemma 4.21 *Let $L_1(\eta, \eta')$ and $L_2(\eta, \eta')$ be two kernels acting on Γ , a simple closed contour. Suppose $L_1(\eta, \eta')$ extends analytically to inside Γ in either the first argument or the second argument and $L_2(\eta, \eta')$ extends analytically to inside Γ in both arguments. Then the Fredholm determinant of $L_1 + L_2$ is equal to the Fredholm determinant of L_1 , ie. $\det(I - \lambda(L_1 + L_2)) = \det(I - \lambda L_1)$.*

Proof By the argument used in lemma 4.20, it is sufficient to show that $Tr((L_1 + L_2))^n = Tr(L_1^n)$ for all $n \in \mathbb{N}$. Assume first that $L_1(\eta, \eta')$ extends analytically to inside Γ in the second argument. The other case follows by symmetry. By analyticity,

$$Tr(L_1 L_2) = \oint_{\Gamma} \oint_{\Gamma} L_1(\eta, z) L_2(z, \eta) dz d\eta = 0$$

and

$$Tr(L_2^2) = \oint_{\Gamma} L_2(\eta, z) L_2(z, \eta) dz d\eta = 0$$

. Also note that $Tr(L_1 L_2) = Tr(L_2 L_1)$ by cyclicity of trace. Consider then $(L_1 + L_2)^n$. For $n = 1$, obviously $Tr(L_1 + L_2) = Tr(L_1)$. For $n = 2$,

$$Tr((L_1 + L_2)^2) = Tr(L_1^2) + Tr(L_2^2) + Tr(L_1 L_2) + Tr(L_2 L_1) = Tr(L_1^2)$$

Assume then $\text{Tr}((L_1 + L_2)^k) = \text{Tr}(L_1^k)$ holds for $k = n - 1$ and consider $k = n$.

$$\text{Tr}((L_1 + L_2)^n) = \text{Tr}((L_1 + L_2)(L_1 + L_2)^{n-1}) = \text{Tr}((L_1 + L_2)(L_1^{n-1} + L_2^{n-2}L_1)) = \text{Tr}(L_1^n)$$

which completes the proof.

Lemma 4.22 *Let Γ be any closed curve around $\eta = 1$ counterclockwise with $\eta = \tau^{-1}$ outside. Then the Fredholm determinant of $K(\xi, \xi')$ acting on \mathcal{C}_R equals the Fredholm determinant of $K_1(\eta, \eta') - K_2(\eta, \eta')$ acting on Γ .*

Proof Since K_2 acting on γ is the same as K acting on \mathcal{C}_R , it is to be shown that the Fredholm determinant of K_2 acting on γ equals the Fredholm determinant of $K_1 - K_2$ acting on Γ . $\tau < 1$ so $\tau^{-k}\eta$ can be assumed to be outside γ for all $k \in \mathbb{N}$. Thus $K_1(\eta, \eta')$ extends analytically to inside γ in both arguments and $K_2(\eta, \eta')$ extends analytically to inside γ in the first argument. Then lemma 4.21 shows that

$$\det(I - \lambda K) = \det(I - \lambda K_2) = \det(I - \lambda(K_2 - K_1)).$$

Consider a deformation $s \mapsto \Gamma_s$ where $\Gamma_0 = \gamma$ and $\Gamma_t = -\Gamma$ for some $t > 0$. Using lemma 4.20 with

$$K_1(\eta, \eta') - K_2(\eta, \eta') = \frac{\phi(\tau\eta) - \phi(\eta')}{\eta' - \tau\eta}$$

shows that γ can be deformed to $-\Gamma$ since any of the singularities at $\eta, \eta' = 1, \tau^{-1}$ are not passed in the deformation Γ_s . This completes the proof since one can then traverse $-\Gamma$ in the opposite direction and obtain the sum of kernels in the claim.

Lemma 4.23 *Assume Γ is a counterclockwise closed curve around $\eta = 1$ with $\eta = \tau^{-1}$ outside. Assume further that Γ is star-shaped with respect to $\eta = 0$. Then*

$$\det(I - \lambda K_1) = \prod_{k=0}^{\infty} (1 - \lambda \tau^k)$$

where K_1 acts on Γ .

Proof Define

$$K_0(\eta, \eta') = \frac{1}{\eta' - \tau\eta}. \quad (4.64)$$

Now $K_1(\eta, \eta') = \phi(\tau\eta)K_0(\eta, \eta')$. Note that $\phi(\tau\eta)$ is non-analytic only at points $\eta = \tau^{-1}$ and $\eta = \tau^{-2}$ which are outside Γ . Thus $\phi(\tau\eta)$ is analytic on $s\Gamma$, $0 < s \leq 1$ and thus by lemma 4.20 the Fredholm determinant of K_1 acting on Γ is the same as the Fredholm determinant of K_1 acting on $s\Gamma$. With a change of variables one can see that this is the same as the Fredholm determinant of

$$K_1^s(\eta, \eta') = sK_1(s\eta, s\eta') = \frac{\phi(s\tau\eta)}{\eta' - \tau\eta}.$$

acting on Γ . Now $\phi(s\tau\eta) = (\frac{1-s\tau^2\eta}{1-s\tau\eta})^x e^{(\frac{1}{1-s\tau\eta} - \frac{1}{1-s\tau^2\eta})t}$. Left-multiplication by $\phi(s\tau\eta)$ now obviously converges to the identity in operator norm as $s \rightarrow 0$ since $\phi(s\tau\eta) \rightarrow 1$ uniformly on Γ . From this it follows that K_1^s converges in trace norm to K_0 as $s \rightarrow 0$. Thus the Fredholm determinant of K_1 acting

on Γ is the same as the Fredholm determinant of K_0 acting on Γ . Since the determinant is determined by the traces, one needs to find $Tr(K_0^n)$. For $n = 2$,

$$K_0^2(\eta, \eta') = \int_{\Gamma} \frac{1}{(z - \tau\eta)(\eta' - \tau z)} dz = \frac{1}{\eta' - \tau^2\eta}. \quad (4.65)$$

Since Γ is star-shaped with respect to the origin, $\tau\eta$ is inside Γ when $\eta \in \Gamma$. For the same reason $\tau^{-1}\eta'$ is outside Γ for $\eta' \in \Gamma$. Now assume $K_0^k(\eta, \eta') = \frac{1}{\eta' - \tau^k\eta}$ holds for $k = n-1$ and show it holds for $k = n$.

$$K_0^n(\eta, \eta') = (K_0 K_0^{n-1})(\eta, \eta') = \int_{\Gamma} \frac{1}{(z - \tau\eta)(\eta' - \tau^{n-1}z)} dz = \frac{1}{\eta' - \tau^n\eta}$$

by same argument as for equation (4.65). Thus $K_0^n(\eta, \eta') = \frac{1}{\eta' - \tau^n\eta}$. From this it follows that

$$Tr K_0^n = \int_{\Gamma} \frac{dz}{z(1 - \tau^n)} = \frac{1}{1 - \tau^n}$$

since Γ is star-shaped with respect to 0. Recall the definition

$$\det(I - \lambda K_0) = e^{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\lambda^n}{n} Tr(K_0^n)}.$$

Now, for small λ ,

$$\log(\det(I - \lambda K_0)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\lambda^n}{n} \frac{1}{1 - \tau^n} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tau^{kn} \lambda^n}{n} = \sum_{k=0}^{\infty} \log(1 - \lambda \tau^k)$$

which implies $\det(I - \lambda K_0) = \prod_{k=0}^{\infty} (1 - \lambda \tau^k)$, completing the proof.

Define $R(\eta, \eta'; \lambda)$ as the kernel of $\lambda(I - \lambda K_1)^{-1} K_1$.

Lemma 4.24 *Assume Γ is a counterclockwise closed curve around $\eta = 1$ with $\eta = \tau^{-1}$ outside. Assume further that Γ is star-shaped with respect to $\eta = 0$. Then, for small enough λ*

$$R(\eta, \eta'; \lambda) = \sum_{k=1}^{\infty} \lambda^k \frac{\phi_n(\tau\eta)}{\eta' - \tau^n\eta}.$$

Proof The series representation of is $R(\eta, \eta'; \lambda) = \sum_{k=1}^{\infty} \lambda^k K_1^k(\eta, \eta')$ for small λ .

$$K_1^2(\eta, \eta') = \int_{\Gamma} \frac{\phi(\tau\eta)\phi(\tau z)}{(z - \tau\eta)(\eta' - \tau z)} dz = \frac{\phi(\tau\eta)\phi(\tau^2\eta)}{\eta' - \tau^2\eta}.$$

Assume $K_1^k(\eta, \eta') = \frac{\phi_k(\tau\eta)}{\eta' - \tau^k\eta}$ holds for $k = n-1$ and show it holds for $k = n$.

$$K_1^n(\eta, \eta') = K_1 K_1^{n-1}(\eta, \eta') = \int_{\Gamma} \frac{\phi(\tau\eta)\phi_{n-1}(\tau\eta)}{(z - \tau\eta)(\eta' - \tau^{n-1}\eta)} dz = \frac{\phi_n(\tau\eta)}{\eta' - \tau^n\eta}.$$

since, again, Γ is star-shaped with respect to $\eta = 0$. This proves the claim. \square

Continuing with the proof of theorem 4.19, note that for any $\lambda \neq \tau^{-k}$, $k \in \mathbb{N}$,

$$\det(I - \lambda K) = \det(I - \lambda(K_1 - K_2)) = \det(I - \lambda K_1) \det(I - \lambda K_2(I - \lambda K_1)^{-1}) = \det(I - \lambda K_1) \det(I + \lambda K_2(I + R)) \quad (4.66)$$

One can now write equation (4.56) again with the help of equation (4.66) as

$$\mathbb{P}(x_m(\frac{t}{\gamma}) \leq x) = \int_{\mathcal{C}_R} \prod_{k=m}^{\infty} (1 - \lambda \tau^k) \det(I + \lambda K_2(I + R)) \frac{d\lambda}{\lambda}. \quad (4.67)$$

With the change of variables $\lambda = \tau^{-m} \mu$ equation (4.67) becomes

$$\mathbb{P}(x_m(\frac{t}{\gamma}) \leq x) = \int_{\mathcal{C}_{R\tau^{-m}}} \prod_{k=0}^{\infty} (1 - \mu \tau^k) \det(I + \tau^{-m} \mu K_2(I + R)) \frac{d\mu}{\mu}. \quad (4.68)$$

Define

$$f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} z^k \quad (4.69)$$

and

$$J(\eta, \eta') = \oint_{\mathcal{C}_{|z|}} \frac{\phi_{\infty}(z)}{\phi_{\infty}(\eta')} \frac{z^m}{(\eta')^{m+1}} \frac{f(\mu, \frac{z}{\eta'})}{z - \eta} dz. \quad (4.70)$$

as the kernel of an operator acting on a origin-centered circle \mathcal{C}_r of radius $r \in (0, 1)$. In (4.70) the contour $\mathcal{C}_{|z|}$ is centered on the origin with radius $1 < |z| < \tau^{-1}r$. Note that equation (4.69) is analytic in the annulus $1 < |z| < \tau^{-1}$.

Lemma 4.25

$$\det(I + \tau^{-m} \mu K_2(I + R)) = \det(I + \mu J)$$

Proof K_1 and K_2 act on the contour Γ which is a closed contour with $\eta = 1$ inside, $\eta = \tau^{-1}$ outside and is star-shaped with respect to the origin. Choose as Γ a circle centered at $\eta = 0$ of radius $r \in (1, \tau^{-1})$. Note that

$$\phi_n(z) = \prod_{k=0}^{n-1} \phi(\tau^k z) = \frac{\prod_{k=0}^{\infty} \phi(\tau^k z)}{\prod_{k=n}^{\infty} \phi(\tau^k z)} = \frac{\phi_{\infty}(z)}{\phi_{\infty}(\tau^n z)}. \quad (4.71)$$

Then

$$K_2 R(\eta, \eta') = \sum_{k=1}^{\infty} \lambda^k \int_{\Gamma} \frac{\phi(z)}{z - \tau \eta} \frac{\phi_k(\tau z)}{\eta' - \tau^k z} dz$$

With the use of identity (4.71)

$$\begin{aligned} K_2 R(\eta, \eta') &= \sum_{k=1}^{\infty} \lambda^k \int_{\Gamma} \phi_{k+1}(z) \frac{1}{(z - \tau \eta)(\eta' - \tau^k z)} dz \\ &= \sum_{k=1}^{\infty} \lambda^k \int_{\Gamma} \frac{\phi_{\infty}(z)}{\phi_{\infty}(\tau^{k+1} z)} \frac{1}{(z - \tau \eta)(\eta' - \tau^k z)} dz. \end{aligned}$$

With the introduction of extra factors from $R(\eta, \eta'; \lambda)$, no new poles are introduced inside the contour if Γ is deformed to \mathcal{C}_z , a circle centered on the origin with radius

$$1 < |z| < \tau^{-1}r.$$

Assuming \mathcal{C}_u is a origin-centered circle of radius $\tau^2 < |u| < \frac{\tau r}{|z|}$

$$\begin{aligned} K_2 R(\eta, \eta') &= \sum_{k=1}^{\infty} \lambda^k \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{z - \tau\eta} \int_{\mathcal{C}_u} \frac{1}{\phi_{\infty}(uz)(\eta' - \frac{uz}{\tau})} \frac{dudz}{u - \tau^{k+1}} \\ &= \sum_{n=0}^{\infty} \frac{\tau^{2n} \lambda}{1 - \tau^n \lambda} \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{z - \tau\eta} \int_{\mathcal{C}_u} \frac{1}{\phi_{\infty}(uz)(\eta' - \frac{uz}{\tau})} \frac{dudz}{u^{n+1}} \end{aligned}$$

where in the last equality $\sum_{n=0}^{\infty} \frac{\tau^{(k+1)n}}{u^{n+1}} = \frac{1}{u - \tau^{k+1}}$ and $\sum_{k=1}^{\infty} \lambda^k \tau^{(k+1)n} = \frac{\lambda \tau^{2n}}{1 - \lambda \tau^n}$. The radius of \mathcal{C}_u is chosen so that only the poles at $u = \tau^{k+1}$ are inside the contour and the series used converge. This can be seen by noting that the other poles of the u-integrand are at $u_1^* = \eta' \frac{\tau}{z}$ and $u_2^* = \frac{1}{z}$.

$$\begin{aligned} |u_1^*| &= |\eta'| \frac{\tau}{|z|} = \frac{\tau r}{|z|} > |u| \\ |u_2^*| &= \frac{1}{|z|} > \frac{\tau r}{|z|} > |u| \end{aligned}$$

since $\tau r < 1$. The next step is to write

$$\frac{\tau^{2k} \lambda}{1 - \tau^k \lambda} = \frac{\tau^k}{1 - \tau^k \lambda} - \tau^k$$

and sum the series separately. As is, both series need not converge, so require a stricter condition

$$\tau < |u| < \frac{\tau r}{|z|}. \quad (4.72)$$

This condition is not necessarily satisfied with the current condition $1 < |z| < \tau^{-1} r$ since from (4.72) one gets $|z| \tau < \tau r$. That is, further require that

$$1 < |z| < r.$$

Define the two now convergent series as

$$\begin{aligned} g_1(\eta, \eta') &= \sum_{k=0}^{\infty} \frac{\tau^k}{1 - \lambda \tau^k} \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{z - \tau\eta} \int_{\mathcal{C}_u} \frac{dudz}{\phi_{\infty}(uz)(\eta' - \frac{uz}{\tau}) u^{k+1}} \\ g_2(\eta, \eta') &= - \sum_{k=0}^{\infty} \tau^k \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{z - \tau\eta} \int_{\mathcal{C}_u} \frac{dudz}{\phi_{\infty}(uz)(\eta' - \frac{uz}{\tau}) u^{k+1}}. \end{aligned}$$

Then

$$K_2 R(\eta, \eta') = g_1(\eta, \eta') + g_2(\eta, \eta').$$

Notice that in $g_1(\eta, \eta')$ the only pole inside the contour \mathcal{C}_u is at $u = 0$, so for $k < 0$ the integrand is analytic in u and thus all terms with negative k vanish. The sum can thus be taken from $k = -\infty$ to $k = \infty$. Do the sum in $g_2(\eta, \eta')$ to obtain

$$- \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{z - \tau\eta} \int_{\mathcal{C}_u} \frac{dudz}{\phi_{\infty}(uz)(\eta' - \frac{uz}{\tau})(u - \tau)} = - \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{\phi_{\infty}(\tau z)} \frac{dz}{(\eta' - z)(z - \tau\eta)} = - \int_{\mathcal{C}_z} \frac{\phi(z)}{(z - \tau\eta)(\eta' - z)} dz$$

since $|u||z| < \tau r < 1$ so that the poles at $u = \frac{\eta'\tau}{z}$ and $u = \frac{1}{z}$ are outside the contour with only the pole at $z = \tau$ inside. Now deform the \mathcal{C}_z contour to a circle of radius $|z| > r$ so that the pole at $z = \eta'$ is passed, obtaining

$$g_2(\eta, \eta') = -\frac{\phi(\eta')}{\eta' - \tau\eta} - \int_{\mathcal{C}_z} \frac{\phi(z)}{(z - \tau\eta)(\eta' - z)} dz = -K_2(\eta, \eta') - \int_{|z|>r} \frac{\phi(z)}{(z - \tau\eta)(\eta' - z)} dz.$$

Then $K_2(I + R)(\eta, \eta')$ can be written

$$K_2(\eta, \eta') + g_1(\eta, \eta') - K_2(\eta, \eta') - \int_{|z|>r} \frac{\phi(z)}{(z - \tau\eta)(\eta' - z)} dz = g_1(\eta, \eta') - \int_{|z|>r} \frac{\phi(z)}{(z - \tau\eta)(\eta' - z)} dz. \quad (4.73)$$

Now do a change of variables $u \rightarrow \frac{u}{z}$ in $g_1(\eta, \eta')$ so that

$$g_1(\eta, \eta') = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \lambda\tau^k} \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{z - \tau\eta} z^k \int_{\mathcal{C}_u} \frac{dudz}{\phi_{\infty}(u)(\eta' - \frac{u}{\tau})u^{k+1}} \quad (4.74)$$

where the contours \mathcal{C}_z and \mathcal{C}_u are both zero-centered circles with radii

$$\begin{aligned} 1 < |z| < r \\ \tau|z| < |u| < \tau r \end{aligned}$$

where $r \in (1, \tau^{-1})$. Since $|\tau\eta| = \tau r < 1 < |z|$, $g_1(\eta, \eta')$ is analytic for $|\eta| \leq r$. The second term in equation (4.73) is analytic for $|\eta|, |\eta'| \leq r$ for the same reason. Using lemma 4.21,

$$\det(I + \tau^{-m}\mu K_2(I + R)) = \det(I + \tau^{-m}\mu g_1)$$

where g_1 is the operator acting on Γ with kernel $g_1(\eta, \eta')$. Equation (4.74) can be simplified using definition (4.69) and reindexing the sum in g_1 by $k \rightarrow k + m$.

$$\sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \lambda\tau^k} \frac{z^k}{u^k} = \tau^m \left(\frac{z}{u}\right)^m \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k\mu} \frac{z^k}{u^k} = \tau^m \left(\frac{z}{u}\right)^m f\left(\mu, \frac{z}{u}\right).$$

Define the kernel of an operator J_0 acting on $\tau\Gamma$ by

$$J_0(\eta, \eta') = \int_{\mathcal{C}_z} \int_{\mathcal{C}_u} \frac{\phi_{\infty}(z)}{\phi_{\infty}(u)} \frac{z^m}{u^{m+1}} \frac{f\left(\mu, \frac{z}{u}\right) dudz}{(z - \eta)(\eta' - u)}. \quad (4.75)$$

Now compare equation (4.75) to

$$\tau^{-m} g_1(\eta, \eta') = \int_{\mathcal{C}_z} \int_{\mathcal{C}_u} \frac{\phi_{\infty}(z)}{\phi_{\infty}(u)} \frac{z^m}{u^{m+1}} \frac{f\left(\mu, \frac{z}{u}\right) dudz}{(z - \tau\eta)(\eta' - u\tau^{-1})}.$$

Deforming $\Gamma \rightarrow \tau\Gamma$ requires $g_1(\eta, \eta') \rightarrow \tau^{-1}g_1(\tau^{-1}\eta, \tau^{-1}\eta')$ for the operator g_1 to remain unchanged. Thus

$$\det(I + \tau^{-m}\mu g_1) = \det(I + \mu J_0).$$

All that is left is to expand the u -contour in equation (4.75) so that $\tau r < |u| < 1$. Only one pole at $u = \eta'$ is passed in the process, and

$$J_0(\eta, \eta') = \int_{\mathcal{C}_z} \int_{\mathcal{C}_u} \frac{\phi_{\infty}(z)}{\phi_{\infty}(u)} \frac{z^m}{u^{m+1}} \frac{f\left(\mu, \frac{z}{u}\right) dudz}{(z - \eta)(\eta' - u)} + \int_{\mathcal{C}_z} \frac{\phi_{\infty}(z)}{\phi_{\infty}(\eta')} \frac{z^m}{(\eta')^{m+1}} \frac{f\left(\mu, \frac{z}{\eta'}\right) dz}{(z - \eta)}. \quad (4.76)$$

where the last integral is the kernel $J(\eta, \eta')$. The only factor in the last term that concerns η is $(z - \eta)$, and thus it is analytic for $|\eta| \leq r$. With the u-contour expanded in the first term, it is analytic for $|\eta|, |\eta'| \leq r$. Then, by lemma 4.21,

$$\det(I + \mu J_0) = \det(I + \mu J)$$

where J is now an operator acting on $\tau\Gamma$ where $|\tau\Gamma| = \tau r$, $1 < |z| < r$ and $r \in (1, \tau^{-1})$ with kernel

$$J(\eta, \eta') = \int_{\mathcal{C}_z} \frac{\phi_\infty(z)}{\phi_\infty(\eta')} \frac{z^m}{(\eta')^{m+1}} \frac{f(\mu, \frac{z}{\eta'}) dz}{(z - \eta)}.$$

The upper bound for $|z|$ can be expanded to $\tau^{-1}r$ without passing any singularities. This proves the claim. \square

The rest of the analysis will consider only the kernel (4.70). Since $\phi_\infty(z) = (1 - z)^{-x} e^{\frac{z}{1-z}t}$, it holds that

$$\phi_\infty(z) z^m = \exp(-x \ln(1 - z) + t \frac{z}{1 - z} + m \ln(z)) = \exp(\kappa(z)).$$

where $\kappa(z)$ was defined as $\kappa(z) = -x \ln(1 - z) + t \frac{z}{1 - z} + m \ln(z)$. The goal is steepest descent analysis, so find the stationary points of $\kappa(z)$ by differentiating it.

$$\kappa'(z) = 0 \Leftrightarrow (m - x)z^2 + (x + t - 2m)z + m = 0. \quad (4.77)$$

Demanding an unique solution to this yields the condition

$$(x + t - 2m)^2 = 4m(m - x).$$

Then set $m = \sigma t$ and $x = c_1 t$. This, and the fact that c_1 should be chosen increasing in t , gives

$$c_1 = 2\sqrt{\sigma} - 1.$$

The unique stationary point is then at $\zeta = -\frac{\sqrt{\sigma}}{1 - \sqrt{\sigma}}$. Add to x a term parametrized by s , $x = c_1 t + c_2 s t^{\frac{1}{3}}$ and write $\phi_\infty(z) z^m = \phi_\infty(\zeta) \zeta^m e^{\psi(z)}$, that is, take out the constant term from the Taylor expansion of $\kappa(z)$ around ζ . Then

$$\psi(z) = -\frac{c_2^3}{3} t (z - \zeta)^3 + c_3 s t^{\frac{1}{3}} (z - \zeta) + O(t(z - \zeta)^4) + O(t^{\frac{1}{3}}(z - \zeta)^2) \quad (4.78)$$

where $c_3 = (1 - \sqrt{\sigma})c_2$. Define

$$\phi(z) = \frac{\kappa(z) - \kappa(\zeta)}{t} = -c_1 \ln\left(\frac{1 - z}{1 - \zeta}\right) + \frac{z}{1 - z} - \frac{\zeta}{1 - \zeta} + \sigma \ln\left(\frac{z}{\zeta}\right). \quad (4.79)$$

The following lemma then guarantees the existence of desired kind of contours for the analysis.

Lemma 4.26 *There exist two disjoint closed curves Γ_η and Γ_z such that*

1. *The part of Γ_η in a neighbourhood N_η of $\eta = \zeta$ is a pair of rays in the directions $\pm \frac{\pi}{3}$ and similarly, the part of Γ_z in a neighbourhood N_z of $z = \zeta$ is a pair of rays from $\zeta - t^{-\frac{1}{3}}$ in the directions $\pm \frac{2\pi}{3}$.*
2. *For some $\delta > 0$, it holds that $\operatorname{Re}(\phi(z)) < -\delta$ on $\Gamma_\zeta \setminus N_\zeta$ and $\operatorname{Re}(\phi(\eta)) > \delta$ on $\Gamma_\eta \setminus N_\eta$.*
3. *The contours Γ and \mathcal{C}_z in the equation (4.70) for the kernel J can be deformed to Γ_η and Γ_z so that the integrand in J remains analytic in all variables during this deformation.*

Proof Near ζ , $\phi(z)$ is dominated by $-c_3^3(z - \zeta)^3$. This and the whole of $\psi(z)$ for $\operatorname{Re}(\psi(z)) = 0$ as seen in figure 1 suggest that there exist three curves on which $\operatorname{Re}(\psi(z))$ vanishes. Name the curves C_i , C_m and C_o , that is, the inside, middle and outside curves, respectively, as in the figure.

The curve consisting of points on which $Re(\psi) = \epsilon$, called C_ϵ between C_i and C_m can be seen in figure 2. Near ζ the curve C_ϵ stays either near C_i or near C_m , the connecting part being a smooth curve between the two. Name the two rays $arg(z - \zeta) = \pm \frac{\pi}{3}$ from ζ to inside C_m as z_ϵ^+ and z_ϵ^- . Then, the curve Γ_η starts from ζ , follows z_ϵ^- until it hits C_ϵ , follows the curve by turning to the right all the way to the intersection of C_ϵ and z_ϵ^+ and then follows z_ϵ^+ back to ζ .

Similarly, the curve consisting of points $Re(\psi) = -\epsilon$ between C_m and C_o , called $C_{-\epsilon}$, can be seen in figure 3. Also the rays $arg(z - \zeta) = \pm \frac{2\pi}{3}$, called $z_{-\epsilon}^-$ and $z_{-\epsilon}^+$ respectively, can be found in the figure. As before, Γ_z is the curve leaving from ζ and making a counterclockwise trip around the outside of C_m through the intersections of $z_{-\epsilon}^-$ and $C_{-\epsilon}$ and $z_{-\epsilon}^+$ and $C_{-\epsilon}$.

Now the first two conditions of the lemma are satisfied by the definition of the contours above. For the third one, simultaneously deform the z and η contours from just outside and inside the unit circle to just outside and inside C_m . This can be done without hitting any poles on the way, and can then be further deformed to the desired contours which are very close to the unit circle. \square

Theorem 4.19 can finally be proven now. From lemmas 4.26 and 4.20 it follows that J can be taken to act on Γ_η and the integral in equation (4.70) being over Γ_ζ . Define the integral operators

$$A : L^2(\Gamma_z) \rightarrow L^2(\Gamma_\eta) \text{ with kernel } A(\eta, z) = \frac{e^{\kappa(z)}}{z - \eta}$$

$$B : L^2(\Gamma_\eta) \rightarrow L^2(\Gamma_z) \text{ with kernel } B(z, \eta) = \frac{\mu f(\mu, \frac{z}{\eta})}{\eta e^{\kappa(\eta)}}$$

so that $\mu J = AB$. By construction both kernels are $O(t^{\frac{1}{3}})$ ignoring the exponentials. This and the fact that the exponentials are small, restrictions to either $z \in \Gamma_z \setminus N_z$ or $\eta \in \Gamma_\eta \setminus N_\eta$ have exponentially small trace norm. Thus J can be restricted to act on $\Gamma_\eta \cap N_\eta$ and the integral taken over $\Gamma_z \cap N_z$ without changing the limit. Without changing the limiting behaviour of the exponentials, one can further restrict z and η to t^{-a} -neighbourhoods of ζ where $a < \frac{1}{3}$ as can be seen from equation (4.78). Then make a change of variables to each of the variables η, η', z

$$x \rightarrow \zeta + \frac{x}{c_3 t^{\frac{1}{3}}}$$

where x is any of the variables. η, η' were originally on rays from ζ to $\zeta + t^{-a} e^{\pm i \frac{\pi}{3}}$, and are now rays from 0 to $c_3 t^{\frac{1}{3}-a} e^{\pm i \frac{\pi}{3}}$. Similarly, z -ray was originally from $\zeta - t^{-\frac{1}{3}}$ to $\zeta - t^{-\frac{1}{3}} + t^{-a} e^{\pm i \frac{2\pi}{3}}$ and is now a pair of rays from $-c_3$ to $-c_3 + c_3 t^{\frac{1}{3}-a} e^{\pm i \frac{2\pi}{3}}$. Near $z = 1$, $f(\mu, z)$ has a simple pole and

$$f(\mu, z) = \frac{\mu^{-1}}{1 - z} + O(1).$$

Since $\frac{z}{\eta}$ is close to 1, it holds that

$$\frac{\mu f(\mu, \frac{z}{\eta})}{\eta} = \frac{1}{\eta - z} + O(t^{-\frac{1}{3}}).$$

Due to the rescaling, the exponentials become (see equation (4.78))

$$e^{\phi(z)} \rightarrow O(e^{-\delta|z|^3})$$

$$e^{-\phi(\eta)} \rightarrow O(e^{-\delta|\eta|^3}).$$

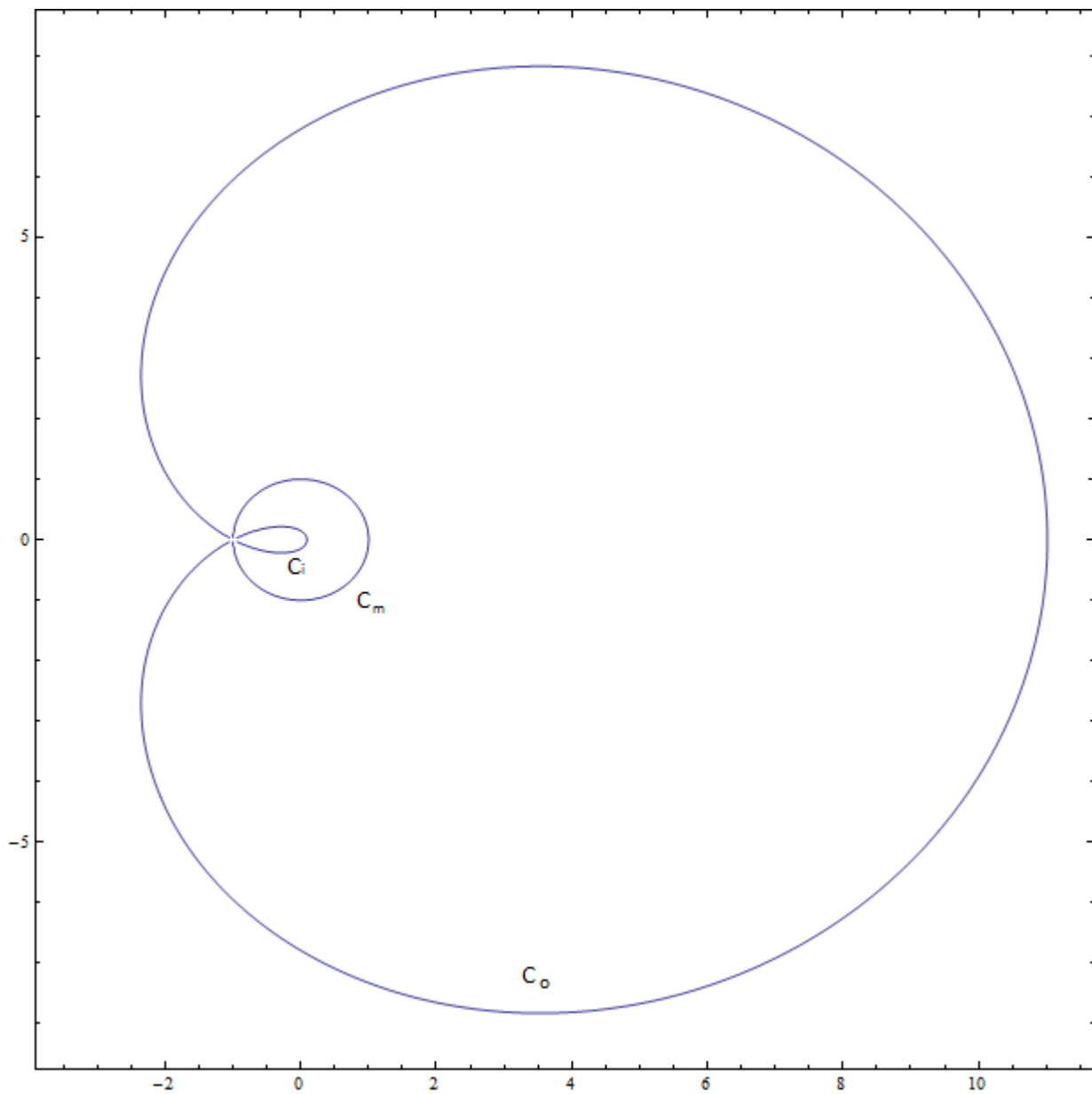


Figure 1: All the curves satisfying $Re(\psi) = 0$. $c_1 = 0$

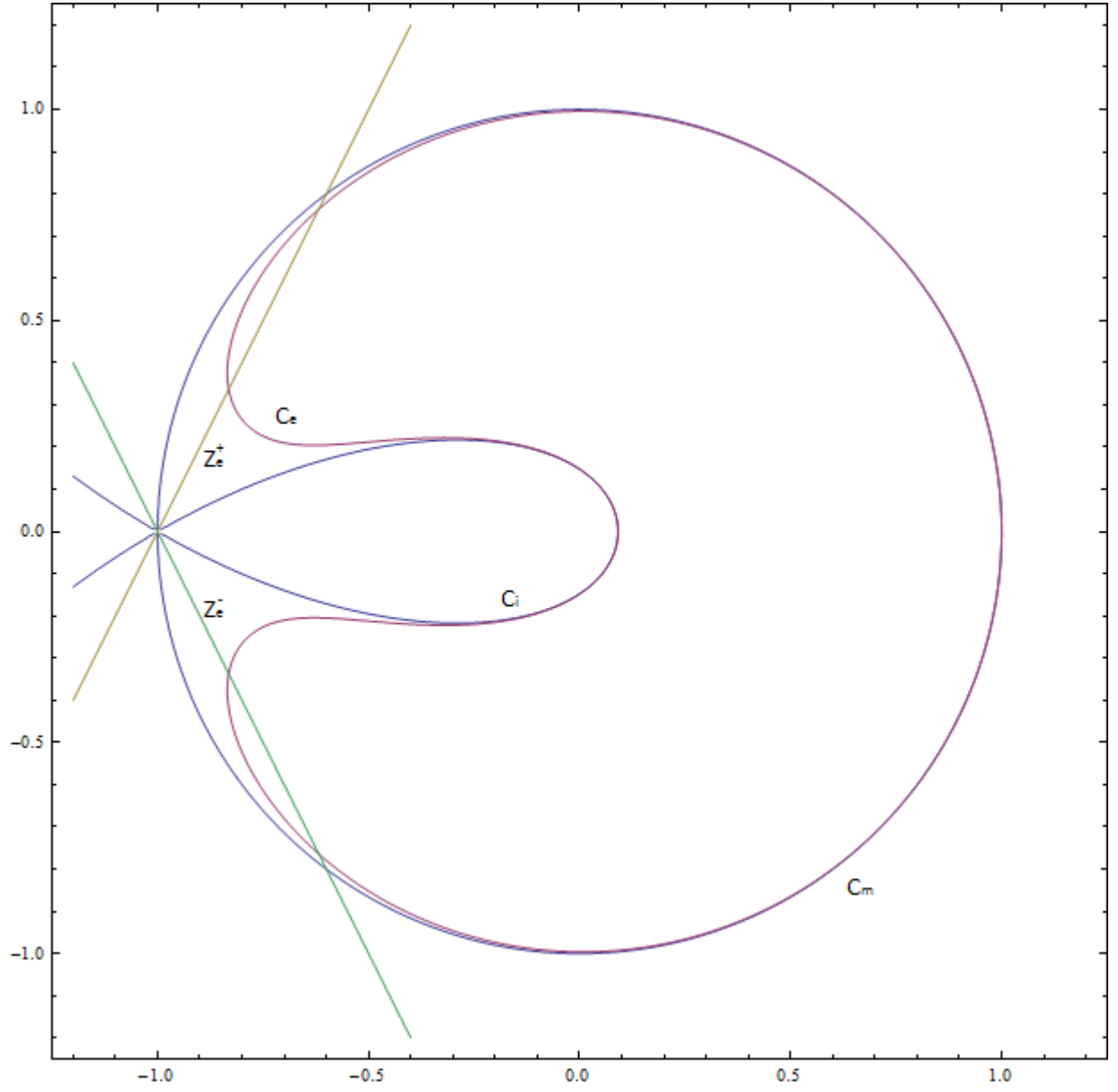


Figure 2: A figure zoomed to include mostly just C_m , C_ϵ and the rays z_ϵ^+ and z_ϵ^- . $c_1 = 0$, $\epsilon = 10^{-3}$

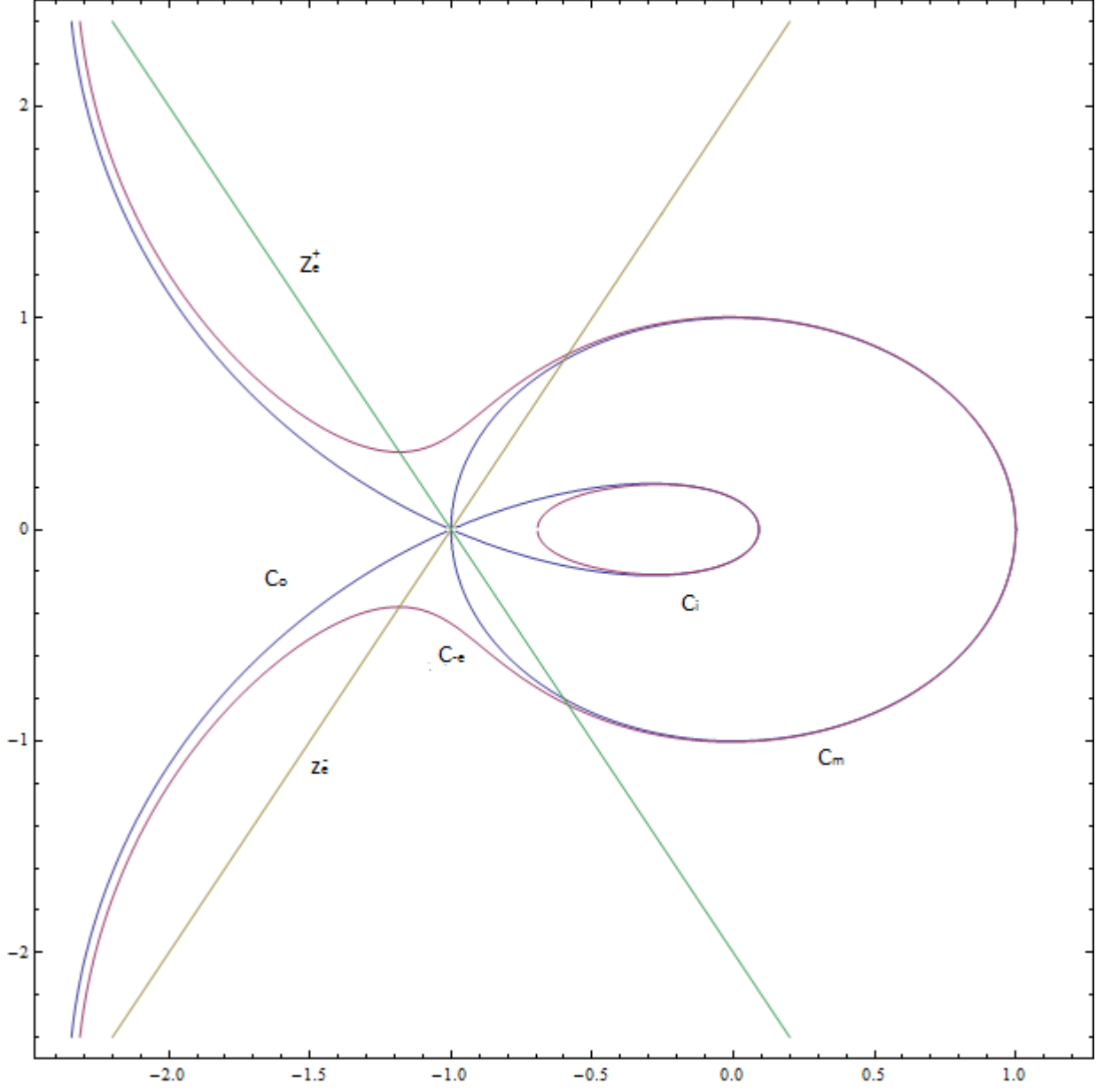


Figure 3: A figure scaled to include a part of C_o leaving from ζ along with the curve $C_{-\epsilon}$ and the rays $z_{-\epsilon}^+$ and $z_{-\epsilon}^-$. $c_1 = 0$, $\epsilon = -10^{-3}$

Thus the operator AB converges in trace norm which guarantees the existence of the limit of the Fredholm determinant of μJ . Restricting a further to $\frac{1}{4} < a < \frac{1}{3}$, the kernels of A and B converge pointwise to

$$\begin{aligned}\lim_{t \rightarrow \infty} A(\eta, z) &= \frac{e^{-\frac{z^3}{3} + sz}}{z - \eta} \\ \lim_{t \rightarrow \infty} B(z, \eta) &= \frac{e^{-\frac{\eta^3}{3} - s\eta}}{\eta - z}.\end{aligned}$$

Thus

$$\lim_{t \rightarrow \infty} \mu J(\eta, \eta') = \int_{\Gamma_z} \frac{e^{-\frac{z^3 - (\eta')^3}{3} + s(z - \eta')}}{(z - \eta)(\eta' - z)} dz \quad (4.80)$$

Define

$$\begin{aligned}A(\eta, z) &= \frac{e^{-\frac{z^3}{3}}}{z - \eta} \\ B(z, x) &= e^{xz} \\ C(x, \eta) &= e^{\frac{\eta^3}{3} - x\eta}.\end{aligned}$$

Then by writing $\frac{e^{s(z - \eta')}}{\eta' - z} = \int_s^\infty e^{x(z - \eta')} dx$, equation (4.80) can be written as the product ABC . By cyclicity, the Fredholm determinant of CAB acting on $L^2(s, \infty)$ is the same as the Fredholm determinant of ABC . CAB has the kernel

$$CAB(x, y) = \int_{\Gamma_z} \int_{\Gamma_\eta} C(x, \eta) A(\eta, z) B(z, y) d\eta dz = \int_{\Gamma_z} \int_{\Gamma_\eta} \frac{e^{\frac{\eta^3 - z^3}{3} + yz - x\eta}}{z - \eta} d\eta dz = -K_{Airy}(x, y).$$

This proves the claim of theorem 4.19. \square

5 Conclusion

It has been shown in this text that there is a direct connection between TASEP and random matrices of a certain type. For ASEP, the connection is not so clear, but looking at the asymptotics for the model yields a huge reward: The Tracy-Widom distribution F_2 can be found in the large scale limit. This kind of connection is not so direct, yet it still exists through the KPZ universality class. As mentioned in the text, certain random matrix ensembles are part of the KPZ universality class (e.g. GUE), as are TASEP and ASEP.

Universality is a way to study a large class of yet unsolved models. Instead of having to go into details of the desired model, one can instead attempt to prove it belongs to a universality class and use that to obtain information about the model. This approach is superior for systems far from equilibrium for which there don't exist methods to analytically solve the desired evolution equations. Non-equilibrium statistical mechanics is very much a current topic of research. Whereas some one-dimensional models are solvable, even two dimensions can shut down most methods used in one-dimensional systems almost

completely. Two dimensional stochastic models have had much success with the advent of Schramm-Loewner Evolution (SLE). This is an example of universality; SLE is a scaling limit of a large number of stochastic planar models. There is, however, virtually no success with three dimensional models.

ASEP has been used to model several kinds of transport and traffic problems [22]. This means, despite the simplicity of the model, ASEP can be applied to a huge number of real world problems, from highway traffic to the protein translation.

The model has been generalized and coupled to several different kinds of other models. These include the Zero Range Process (ZRP), for which the particle jump rate for each site depends on the number of particles next to it and q-TASEP for which the jump rate depends on the number of empty sites directly behind the particle. Albeit the similarity of the models, there are not many similar results for ZRP. However, even for ASEP itself, the results are still fairly limited. The asymptotics in this paper only make sense for a certain initial condition, and a determinantal form for the distribution function has been found only for a few different initial configurations.

There is no doubt interacting particle systems research stays a hot topic for a good while; There are not relatively many useful results on such models, yet the range of potential applications is huge.

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